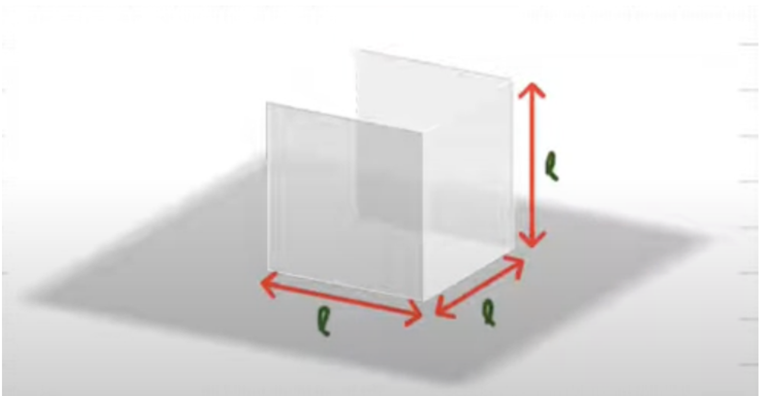
**MAT1475, Fall 2024** **Exam #3 (in-class)**
**Instructor: Suman Ganguli** **Monday, December 16** **Name:** ______________________

| Question: | 1 | 2 | 3 | Total |
|---|---|---|---|---|
| Points: | 5 | 10 | 10 | 25 |
| Score: | | | | |

1. (5 points) Suppose we are observing a melting ice cube. Assuming it maintains the shape of a cube as it melts, both its volume $V$ and its edge length $l$ are decreasing over time.




(a) Clearly $V(t) = l(t)^3$. Use the Chain Rule to express *the rate of change* of the volume $V(t)$ in terms of $l(t)$ and the rate of change of $l(t)$ (i.e., calculate $\frac{dV}{dt}$ in terms of $l(t)$ and $\frac{dl}{dt}$):

**Solution:**

$$\frac{dV}{dt} = 3l^2 \cdot \frac{dl}{dt}$$

(b) Suppose we estimate that the edge length $l$ is decreasing at a constant rate of 2 cm/min. Find the rate of change of the volume when the edge length $l$ is 5 cm. Include units in your calculation.

**Solution:** The given rate at which the edge length is decreasing is the value of $\frac{dl}{dt}$, so $\frac{dl}{dt} = -2$ cm/min. Then:

$$\frac{dV}{dt}|_{l=5cm} = 3 \cdot (5cm)^2 \cdot (-2\,cm/min) = -150\,cm^3/min$$

2. (10 points) Recall that the "linearization" (or "linear approximation") $L(x)$ for a given function $f(x)$ at a point $x = x_0$ is just the equation of the tangent line at $x = x_0$.

(a) Find the linear approximation $L(x)$ for $f(x) = e^{2x}$ at $x_0 = 0$:

**Solution:**

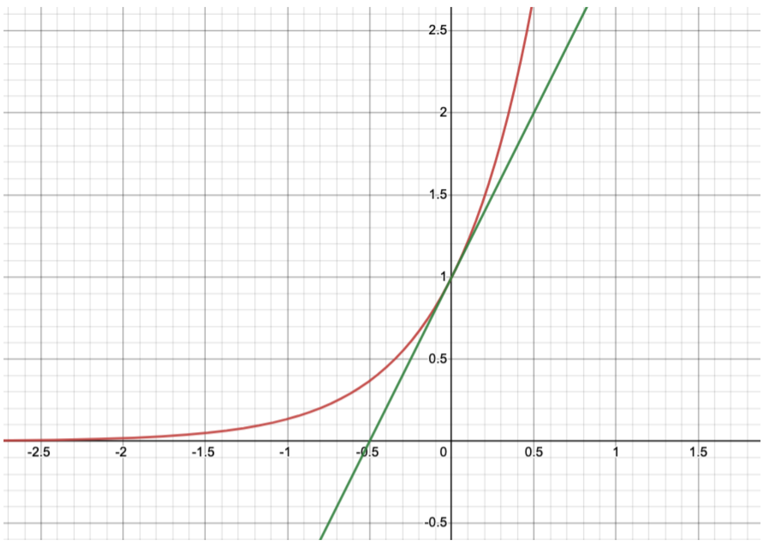
$$f'(x) = 2e^{2x}$$

$$f'(0) = 2e^0 = 2$$

$$f(0) = e^0 = 1$$

$$L(x) = 1 + 2(x - 0) = 2x + 1$$

(b) Shown below is the graph of $f(x) = e^{2x}$. Add the linear approximation (i.e., tangent line) at $x_0 = 0$ to the graph:




(c) Use the linear approximation from part (a) to estimate $e^{0.02}$ and $e^{0.2}$, i.e., calculate $L(0.01)$ and $L(0.01)$:

**Solution:**

$$e^{0.02} = f(0.01) \approx L(0.01) = 2(0.01) + 1 = 1.02$$

$$e^{0.2} = f(0.1) \approx L(0.1) = 2(0.1) + 1 = 1.2$$

3. (10 points) Consider the function $f(x) = x^3 - 3x^2 - 9x + 4$

(a) The first derivative of $f(x)$ is given below. Find the second derivative:

**Solution:**

$$f'(x) = 3x^2 - 6x - 9$$

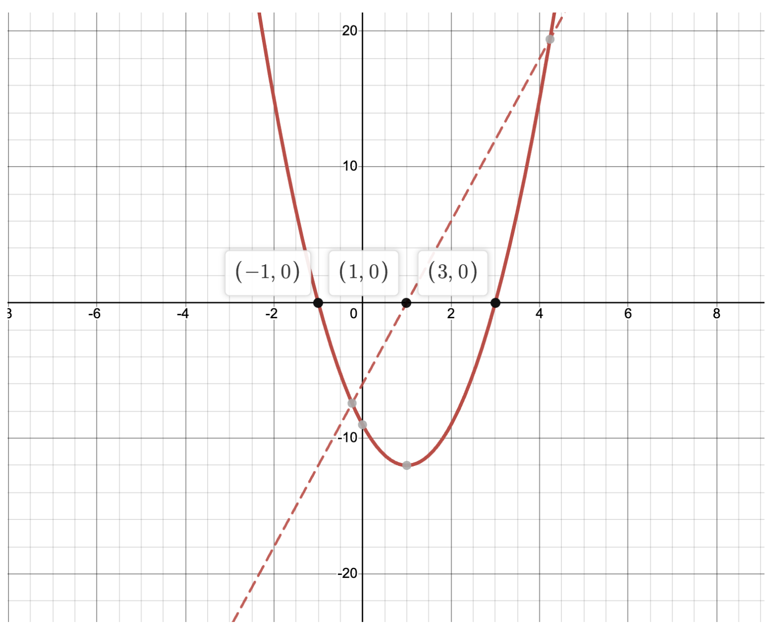
$$f''(x) = 6x - 6$$

(b) Find the critical points and inflection points of $f$, i.e., solve for the values $x$ such that $f'(x) = 0$ and $f''(x) = 0$:
(Hint: Use the factored form of $f'(x)$ given above to solve $f'(x) = 0$.)

**Solution:** Since $f'(x) = 3(x-3)(x+1)$, the critical points occur at $x = 3$ and $x = -1$.
For the inflection point: $f''(x) = 6x - 6 = 0 \implies x = 1$

(c) Sketch rough graphs of $y = f'(x)$ and $y = f''(x)$. In particular, plot the $x$-intercepts of these graphs, using your results from (b).

**Solution:** We use the points we solved for above: $y = f'(x)$ is a parabola with $x$-intercepts at $x = -1$ and $x = 3$
$y = f''(x) = 6x - 6$ is just a straight line with $x$-intercept at $x = 1$




(d) Use your graphs above to find the intervals on which:

**Solution:** We observe where the graphs of $y = f'(x)$ and $y = f''(x)$ are above/below the $x$-axis:

$f'(x) > 0 : (-\infty, -1) \cup (3, \infty)$

$f'(x) < 0 : (-1, 3)$

$f''(x) > 0 : (1, \infty)$

$f''(x) < 0 : (-\infty, 1)$

(e) Use part (d) to write down the intervals on which the graph $y = f(x)$ is:

**Solution:**

- increasing concave up: $(3, \infty)$
- increasing concave down: $(-\infty, -1)$
- decreasing concave up: $(1, 3)$
- decreasing concave down: $(-1, 1)$

(f) Sketch the graph of $y = f(x)$. Label the critical points on the graph, and indicate whether each is a local maximum or a local minimum. Also label the $y$-intercept and the inflection point.
(Hint: since $f(-1) = 9$, $f(1) = -7$, and $f(3) = -23$, the points $(-1, 9), (1, 7)$ and $(3, -23)$ are on the graph.)