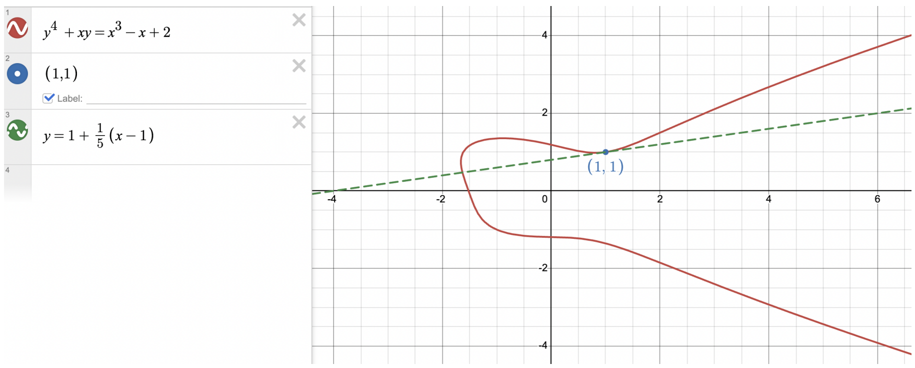
MAT1475, Fall 2024 **Exam #3 (take-home)**
Instructor: Suman Ganguli **Due: Monday, December 16** **Name:** ______________________

Question:	1	2	3	4	5	Total
Points:	5	5	5	5	5	25
Score:						

1. (5 points) Show below is the graph of the algebraic curve defined by the equation

$$y^4 + xy = x^3 - x + 2$$



(a) Verify algebraically that the point $(1,1)$ is on the curve (i.e., show that $x = 1, y = 1$ satisfies the equation):

Solution: Substituting $x = 1, y = 1$into both sides of the equation:

$$1^4 + (1)(1) \stackrel{?}{=} 1^3 - 1 + 2$$

$$1 + 1 \stackrel{?}{=} 1 - 1 + 2$$

$$2 = 2$$

(b) Sketch the tangent line to the curve at the point $(1,1)$ on the graph above.

(c) Use implicit differentiation to find $\frac{dy}{dx}$:

Solution:
We use the y' notation below instead of $\frac{dy}{dx}$. Note that we have to use the product rule in order to differentiate xy:

$$4y^3y' + (y + xy') = 3x^2 - 1$$

$$y'(4y^3 + x) = 3x^2 - 1 - y$$

$$y' = \frac{3x^2 - 1 - y}{4y^3 + x}$$

(d) Write the equation of the tangent line to the curve at the point $(1,1)$:

Solution: The slope is

$$m = y'(1,1) = \frac{3(1^2) - 1 - 1}{4(1^3) + 1} = \frac{1}{5}$$

So the equation of the tangent line is

$$y = 1 + \frac{1}{5}(x - 1)$$

2. (5 points) Suppose an oil tanker starts leaking oil, creating an expanding circular oil spill on the water.

(a) Draw a picture illustrating the situation, and label the radius of the oil spill with the variable $r(t)$:

> **Solution:** Just draw a circle, with maybe an oil tanker at the center, and label the radius $r(t)$. This video of a similar related rates exercise includes a picture:
> `https://www.youtube.com/watch?app=desktop&v=nh56AppTg6U`

(b) Express the surface area $A(t)$ of the oil spill as a function of its radius $r(t)$ (i.e., just the area of the circle!):

> **Solution:** Area of a circle as a function of its radius: $A(t) = \pi r(t)^2$

(c) Clearly, if oil spill is expanding, both the surface area and the radius are increasing with time. Use the Chain Rule to express *the rate of change* of the surface area $A(t)$ with respect to time, in terms of the radius $r(t)$ and the rate of change of the radius with respect to time (i.e., calculate $\frac{dA}{dt}$ in terms of $r(t)$ and $\frac{dr}{dt}$):

> **Solution:** We apply the Chain Rule in order to differentiate $r(t)^2$ with respect to t:
>
> $$\frac{dA}{dt} = \pi \cdot (2 \cdot r(t) \cdot \frac{dr}{dt}) = 2\pi \cdot r(t) \cdot \frac{dr}{dt}$$

(d) Now answer the questions in Problem 1 of the WebWork set "Exam 3," showing all your calculations below. Include units in your calculations:

> **Solution:** WebWork randomizes the numbers in a given exercise. Here is an example:
> "The radius of a circular oil slick expands at a rate of 4 m/min."
> The rate at which the radius is increasing is the value of $\frac{dr}{dt}$. So in this case: $\frac{dr}{dt} = 4m/min$.
> "How fast is the area of the oil slick increasing when the radius is 27 m?"
> We need to calculate $\frac{dA}{dt}$ when $r = 27$; we substitute into the "differential equation" we found in (b):
>
> $$\frac{dA}{dt}|_{r=27m} = 2\pi \cdot 27(m) \cdot 4(m/min) = 216\pi \, m^2/min$$
>
> "If the radius is 0 at time $t = 0$, how fast is the area increasing after 2 mins?" We need to find the radius after 2 mins. Since $r(0) = 0$, and since the radius increases at 4 m/min, $r(2) = 0m + 2(mins) \cdot 4(m/min) = 8m$, i.e., the radius is $8m$ after 2 mins. We substitute this value of r into the equation:
>
> $$\frac{dA}{dt}|_{r=8m} = 2\pi \cdot 8(m) \cdot 4(m/min) = 64\pi \, m^2/min$$

Entered	Answer Preview	Result
678.584013175395	216π	correct
201.061929829747	64π	correct

All of the answers above are correct.

The radius of a circular oil slick expands at a rate of 4 m/min.

(a) How fast is the area of the oil slick increasing when the radius is 27 m?

$\frac{dA}{dt} =$ $216 \cdot \pi$ m^2/min

(b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 2 mins?

$\frac{dA}{dt} =$ 64π m^2/min

3. (5 points) Recall that the "linearization" (or "(local) linear approximation") $L(x)$ for a given function $f(x)$ at a point $x = x_0$ is just the equation of the tangent line for the graph $y = f(x)$ at the point $(x_0, f(x_0))$.

(a) Look at Problem 2 in the WebWork set "Exam 3." Write down the function $f(x)$ and the value of x_0 given in your WebWork exercise, and then evaluate $f(x_0)$:

Solution: Every WebWork exercise had slightly different functions $f(x)$ and values x_0. For these solutions, we will use $f(x) = \sqrt{5+x}$ and $x_0 = 4$, in which case:

$$f(x_0) = f(4) = \sqrt{5+4} = \sqrt{9} = 3$$

(b) As asked in part (a) of the WebWork exercise, find the linear approximation $L(x)$ for the given function $f(x)$ at the given value of x_0, by writing down the following:

Solution: Continuing with $f(x) = \sqrt{5+x}$ and $x_0 = 4$:

$$f'(x) = \frac{1}{2\sqrt{x+9}}$$

$$f'(0) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$L(x) = 3 + \frac{1}{6}(x-4)$$

(c) Use the linear approximation from part (a) to estimate the values of $f(x)$ asked for in parts (b) and (c) of the WebWork exercise. Show your calculations here (you should not need a calculator for this!):

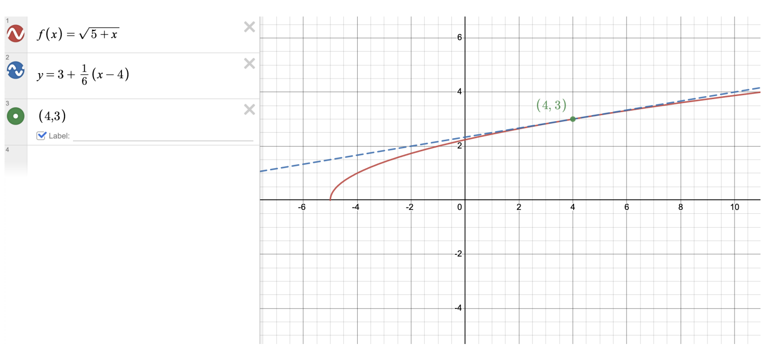
Solution: Suppose we are asked to use the linear approximation to approximate $\sqrt{8.9}$ and $\sqrt{9.1}$. In order to use our linear approximation to $f(x) = \sqrt{5+x}$, we have to express these values in terms of $f(x)$: note that $\sqrt{9.1} = \sqrt{5+4.1} = f(4.1)$. So we calculate $L(4.1)$:

$$\sqrt{9.1} = \sqrt{5+4.1} = f(4.1) \approx L(4.1) = 3 + \frac{1}{6}(4.1-4) = 3 + \frac{1}{6}(0.1) = 3 + \frac{1}{60}$$

Similarly:

$$\sqrt{8.9} = \sqrt{5+3.9} = f(3.9) \approx L(3.9) = 3 + \frac{1}{6}(3.9-4) = 3 + \frac{1}{6}(-0.1) = 3 - \frac{1}{60}$$

(d) Sketch the graph of $f(x)$, and sketch the linear approximation (i.e., the tangent line) at x_0 which you found in (a):



(e) Are the estimated values in part (b) overestimates or underestimates for the exact values for $f(x)$, i.e., is $L(x) > f(x)$ or is $L(x) < f(x)$? Give a brief explanation in terms of the graphs of $f(x)$ and $L(x)$ you sketched above.

Solution: We can see from the graph that the linear approximation (i.e., the tangent line) clearly sits above the curve–in other words, the y-values on the line are greater than the y-values on the curve (for any given value of x): $L(x) > f(x)$! Thus, the estimated values are overestimates.

4. (5 points) Read Problem 3 of the WebWork set, which describes a situation where a landscape architect wants to enclose a rectangular garden on one side by a brick wall and on the other three sides by metal fencing.

(a) Using the variables x and y and the costs per foot for a brick wall and for metal fencing given in your WebWork, write down an expression for the total cost C in terms of x and y. Also draw a sketch of the garden, labeling it with the variables.

Solution:
Here is a sample WebWork exercise: "A landscape architect wished to enclose a rectangular garden on one side by a brick wall costing \$50/ft and on the other three sides by a metal fence costing \$20/ft. If the area of the garden is 122 square feet, find the dimensions of the garden that minimize the cost."
With these numbers, and if we let $x =$ length of brick wall side and $y =$ length of adjacent side, then the total cost is:

$$C(x, y) = 50x + 20y + 20y + 20x = 70x + 40y$$

(Obviously the brick wall will cost $50x$; the other 3 terms represent the cost of the 2 adjacent metal-fence sides of length y, and the metal-fence side opposite the brick side (so also of length x). Note that we have written the cost C here as a function of two variables, x and y, and that the cost C is the quantity we are asked to minimize.)

(b) Write down the "constraint equation," by expressing the given area of the garden in terms x and y.

Solution: Clearly the area of the rectangular garden is xy; in this example, we are told the area is 122 square feet. Thus, the constraint equation is: $xy = 122$

(c) As asked in the WebWork exercise, solve for the dimensions of the garden that minimize the cost C.

Solution: Since the constraint in this exercise is $xy = 122$, we get $y = \dfrac{122}{x}$. Substituting this for y in $C(x, y)$, we get cost as a function of x:

$$C(x) = 70x + 40\left(\frac{122}{x}\right) = 70x + \frac{4880}{x}$$

Then:

$$C'(x) = 70 - \frac{488}{x^2} = \frac{70x^2 - 4880}{x^2}$$

So the critical points of $C(x)$ occur when $70x^2 - 4880 = 0$, i.e., for $x^2 = \dfrac{4880}{70}$. Hence, the maximal area occurs for $x = \sqrt{\dfrac{4880}{70}} \approx 8.35$ (feet)

To solve for the corresponding value of y, we use $y = \dfrac{122}{x}$, and so $y \approx \dfrac{122}{8.35} = 14.61$ (feet)

Entered	Answer Preview	Result
8.34951	$\sqrt{\frac{4880}{70}}$	correct
14.6116	$\frac{122}{\sqrt{\frac{4880}{70}}}$	correct

All of the answers above are correct.

A landscape architect wished to enclose a rectangular garden on one side by a brick wall costing \$50/ft and on the other three sides by a metal fence costing \$20/ft. If the area of the garden is 122 square feet, find the dimensions of the garden that minimize the cost.

5. (5 points) Consider the function $f(x) = \frac{1}{3}x^3 - 3x^2 + 8x + 4$

(a) Find the first and second derivatives of $f(x)$:

Solution: $f'(x) = x^2 - 6x + 8$, $f''(x) = 2x - 6$

(b) Find the critical points of f, i.e., solve for the values x such that $f'(x) = 0$:

Solution: $f'(x) = x^2 - 6x + 8 = (x-4)(x-2) = 0 \Longrightarrow x = 2, 4$

(c) For what values of x is $f'(x) > 0$ and for what values of x is $f'(x) < 0$? Show or explain how you solve for these intervals. What do these intervals represent in terms of the shape of the graph of $f(x)$?

Solution: Since $f'(x)$ is a quadratic with a positive leading term with roots at $x = 2$ and $x = 4$, we can conclude that $f'(x) > 0$ on $(-\infty, 2) \cup (4, \infty)$, meaning the graph is increasing on these intervals. On the other hand $f'(x) < 0$ on $(2, 4)$, and so the graph is decreasing on that interval.

It can help to draw a rough sketch of the graph, as I asked you to do on the in-class exercise.

(d) For what values of x is $f''(x) > 0$ and for what values of x is $f''(x) < 0$? Again, show or explain how you solve for these intervals, and explain what these intervals represent in terms of the shape of the graph of $f(x)$:

Solution: Since $f''(x) = 2x - 6$, which is a linear function with positive slope and x-intercept at $x = 3$, it is clear that $f''(x) > 0$ for all x in $(3, \infty)$ (and so the graph is concave up there) while $f''(x) < 0$ on $(-\infty, 3)$, where the graph is concave down. Hence, $x = 3$ is an inflection point.

(e) Sketch the graph of $y = f(x)$. Label the critical point(s) on the graph, and indicate whether each is a local maximum or a local minimum. Also label the y-intercept and any inflection points.