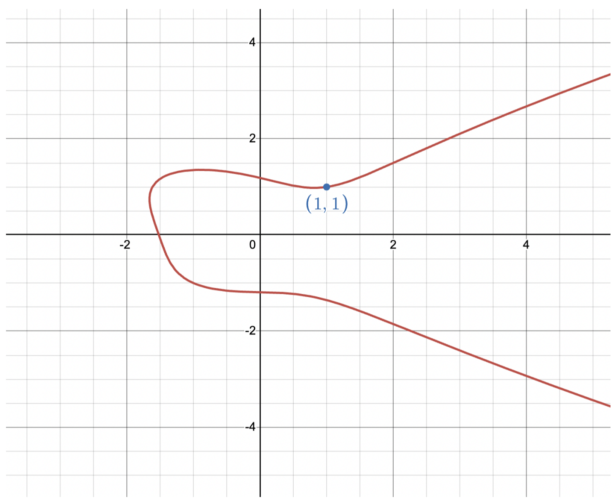
MAT1475, Fall 2024 **Exam #3 (take-home)**
Instructor: Suman Ganguli **Due: Monday, December 16** **Name:** ____________________

Question:	1	2	3	4	5	Total
Points:	5	5	5	5	5	25
Score:						

In order to receive full credit, you must **show all your work**, and write out your solutions in a clear and organized manner. **Please work on this exam individually.** You can (and should) consult resources such as your class notes, the textbook, the Final Exam Review solutions, etc. in order to work through these exercises.

1. (5 points) Show below is the graph of the algebraic curve defined by the equation

$$y^4 + xy = x^3 - x + 2$$



(a) Verify algebraically that the point $(1, 1)$ is on the curve (i.e., show that $x = 1, y = 1$ satisfies the equation):

(b) Sketch the tangent line to the curve at the point $(1, 1)$ on the graph above.

(c) Use implicit differentiation to find $\frac{dy}{dx}$:

(d) Write the equation of the tangent line to the curve at the point $(1, 1)$:

2. (5 points) Suppose an oil tanker starts leaking oil, creating an expanding circular oil spill on the water.

   (a) Draw a picture illustrating the situation, and label the radius of the oil spill with the variable $r(t)$:

   (b) Express the surface area $A(t)$ of the oil spill as a function of its radius $r(t)$ (i.e., just the area of the circle!):

   (c) Clearly, if oil spill is expanding, both the surface area and the radius are increasing with time. Use the Chain Rule to express *the rate of change* of the surface area $A(t)$ with respect to time, in terms of the radius $r(t)$ and the rate of change of the radius with respect to time (i.e., calculate $\frac{dA}{dt}$ in terms of $r(t)$ and $\frac{dr}{dt}$):

   $$\frac{dA}{dt} =$$

   (d) Now answer the questions in Problem 1 of the WebWork set "Exam 3," showing all your calculations below. Include units in your calculations:

3. (5 points) Recall that the "linearization" (or "(local) linear approximation") $L(x)$ for a given function $f(x)$ at a point $x = x_0$ is just the equation of the tangent line for the graph $y = f(x)$ at the point $(x_0, f(x_0))$.

(a) Look at Problem 2 in the WebWork set "Exam 3." Write down the function $f(x)$ and the value of x_0 given in your WebWork exercise, and then evaluate $f(x_0)$:

$f(x) =$

$x_0 =$

$f(x_0) =$

(b) As asked in part (a) of the WebWork exercise, find the linear approximation $L(x)$ for the given function $f(x)$ at the given value of x_0, by writing down the following:

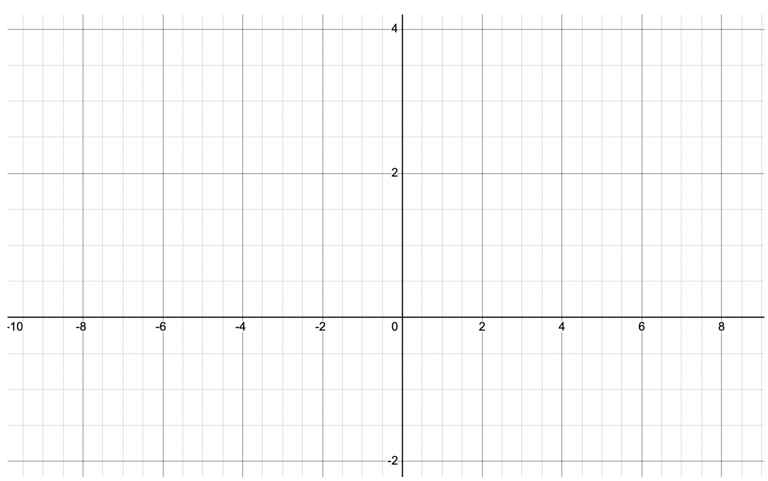
$f'(x) =$

$f'(x_0) =$

$L(x) = f(x_0) + f'(x_0)(x - x_0) =$

(c) Use the linear approximation from part (a) to estimate the values of $f(x)$ asked for in parts (b) and (c) of the WebWork exercise. Show your calculations here (you should not need a calculator for this!):

(d) Sketch the graph of $f(x)$, and sketch the linear approximation (i.e., the tangent line) at x_0 which you found in (a):



(e) Are the estimated values in part (b) overestimates or underestimates for the exact values for $f(x)$, i.e., is $L(x) > f(x)$ or is $L(x) < f(x)$? Give a brief explanation in terms of the graphs of $f(x)$ and $L(x)$ you sketched above.

4. (5 points) Read Problem 3 of the WebWork set, which describes a situation where a landscape architect wants to enclose a rectangular garden on one side by a brick wall and on the other three sides by metal fencing.

   (a) Using the variables x and y and the costs per foot for a brick wall and for metal fencing given in your WebWork, write down an expression for the total cost C in terms of x and y. Also draw a sketch of the garden, labeling it with the variables.

   (b) Write down the "constraint equation," by expressing the given area of the garden in terms x and y.

   (c) As asked in the WebWork exercise, solve for the dimensions of the garden that minimize the cost C.

   Hint: use the "constraint equation" to solve for y in terms of x, and substitute that into C in order to express the cost C as a function of only x. Then find the minimum of $C(x)$ by finding its critical point(s).

5. (5 points) Consider the function $f(x) = \frac{1}{3}x^3 - 3x^2 + 8x + 4$

(a) Find the first and second derivatives of $f(x)$:

$f'(x) =$

$f''(x) =$

(b) Find the critical points of f, i.e., solve for the values x such that $f'(x) = 0$:

(c) For what values of x is $f'(x) > 0$ and for what values of x is $f'(x) < 0$? Show or explain how you solve for these intervals. What do these intervals represent in terms of the shape of the graph of $f(x)$?

(d) For what values of x is $f''(x) > 0$ and for what values of x is $f''(x) < 0$? Again, show or explain how you solve for these intervals, and explain what these intervals represent in terms of the shape of the graph of $f(x)$:

(e) Sketch the graph of $y = f(x)$. Label the critical point(s) on the graph, and indicate whether each is a local maximum or a local minimum. Also label the y-intercept and any inflection points.