MAT1275/D225, Spring 2024 **Exam #3 (take-home exercises)**
Instructor: Suman Ganguli **Due: Monday, May 20** **Name:** ____________________

Question:	1	2	3	4	5	Total
Points:	5	5	5	5	5	25
Score:						

In order to receive full credit, you must **show all the algebra needed to arrive at the solutions** and you should simplify your answers.

1. (5 points) Consider the following equation of a circle: $x^2 + y^2 + 4x - 6y - 36 = 0$

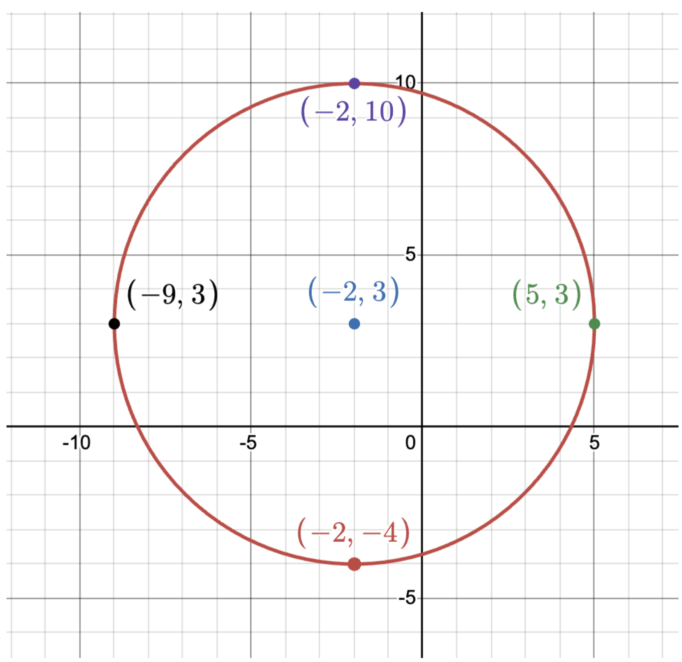
   Write the equation of the circle given below in standard form, and use it to identify the center and radius of the circle. (Recall that the standard form of the equation of a circle centered at (h, k) with radius r is $(x - h)^2 + (y - k)^2 = r^2$. You should show all the steps of completing the square on both the $x-$ and y-terms.)

   Solution: Completing the square on the x-terms:

   $$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 36 + 4 + 9 \implies (x + 2)^2 + (y - 3)^2 = 49$$

   Hence, the circle is centered at $(-2, 3)$, with radius $r = \sqrt{49} = 7$, and so four points on the circle are: $(5, 3), (-2, 10), (-9, 3), (-2, -4)$.

   Use the center and radius to identify four points on the circle, and use these to sketch the graph of the circle. **Label the center and the four points on the circle on your graph with each point's (x, y) coordinates.**



2. (5 points) Solve the following system of equations:

$$\begin{aligned} 3x + y &= 5 \qquad (1) \\ x^2 - 4y &= -40 \qquad (2) \end{aligned}$$

(Hint: start by using equation (1) to solve for y in terms of x, then substitute into equation (2) in order to eliminate y. Then solve the resulting quadratic equation for x.)

Solution: From equations (1): $3x + y = 5 \Rightarrow y = 5 - 3x$

Substituting into equation (2): $x^2 - 4y = -40 \Rightarrow x^2 - 4(5 - 3x) = -40 \Rightarrow x^2 - 20 + 12x = -40 \Rightarrow x^2 + 12x + 20 = 0$

We can solve this quadratic formula for x by factoring: $(x + 10)(x + 2) = 0 \Rightarrow x = -2, x = -10$

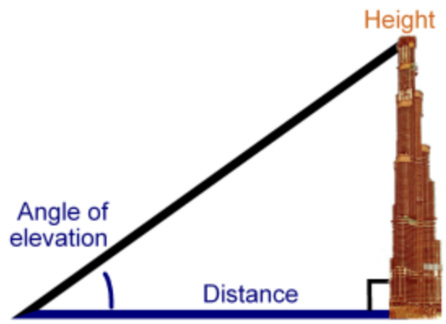
We find the corresponding y-values using $y = 5 - 3x$:

$x = -2 \Rightarrow y = 5 - 3(-2) = 5 + 6 = 11$

$x = -10 \Rightarrow y = 5 - 3(-10) = 5 + 30 = 35$

So the solutions are $(-2, 11)$ and $(-10, 35)$.

3. (5 points) Use the following diagram to show how trigonometry can be used to estimate the height of a building:



In particular, solve for the building height h in terms of the angle of elevation θ and the horizontal distance d. How could the angle θ and distance d be measured in the real world?

Solution: Since the building height h is "opposite" the angle of elevation θ and the horizontal distance d is "adjacent" to θ:

$$\tan\theta = \frac{h}{d} \Longrightarrow h = d\tan\theta$$

The horizontal distance d could be measure with a tape measure (or in the case of a longer distance, a measuring "wheel"). The angle could be measure with a protractor–although in practice, surveyors, engineers, etc. use a special tool called a "theodolite" for measuring angles:

https://en.wikipedia.org/wiki/Theodolite)

4. (5 points) Use the given information to find the values of the trigonometric functions for the angle θ such that:

$$\tan\theta = -\frac{3}{4} \text{ and } \cos\theta < 0$$

(a) Determine which quadrant θ is in by filling in the blanks:

Solution:

- $\tan\theta < 0$ in Quadrant 2 and Quadrant 4
- $\cos\theta < 0$ in Quadrant 2 and Quadrant 3
- Therefore, θ must be in Quadrant 2

(b) Draw a triangle corresponding to θ in the correct quadrant. Find the values of x, y, and r corresponding to the angle θ and label the sides of the triangle with those values.

Solution: Since $\tan\theta = \frac{y}{x} = -\frac{3}{4}$, and because θ is in Quadrant 2 (meaning $x < 0$ and $y > 0$), we conclude that $y = 3$ and $x = -4$.
We solve for r using the Pythagorean Theorem $r^2 = x^2 + y^2$:

$$r^2 = (-4)^2 + (3)^2 = 16 + 9 = 25 \Longrightarrow r = 5$$

(c) Finally, use the following definitions to find the values of the five other trig functions:

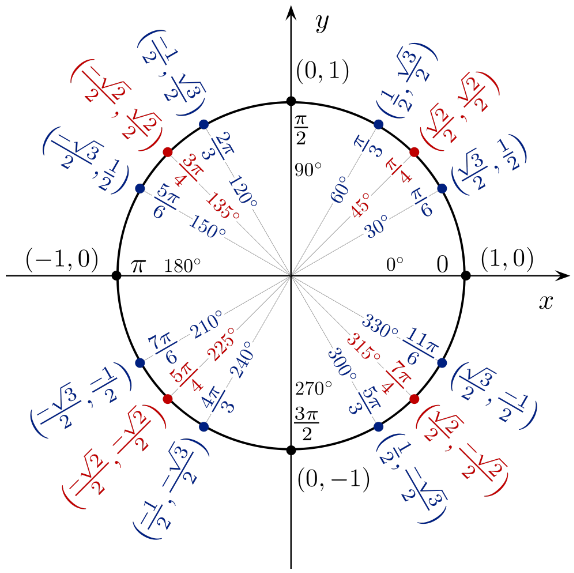
Solution: Since $x = -4, y = 3, r = 5$, the values of the trig functions are:

$$\sin\theta = \frac{3}{5} \qquad \csc\theta = \frac{5}{3}$$

$$\cos\theta = -\frac{4}{5} \qquad \sec\theta = -\frac{5}{4}$$

$$\tan\theta = -\frac{3}{4} \qquad \cot\theta = -\frac{4}{3}$$

5. (5 points) Use the unit circle to find the solutions θ of the following equations, for $0 \leq \theta < 2\pi$. For each equation, circle the $x-$ and/or $y-$coordinates on the unit circle and the corresponding angles that you use to find the solutions.



(a) $8\sin\theta - 4 = 0$

Solution: $8\sin\theta - 4 = 0 \iff \sin\theta = \dfrac{1}{2}$, so we look for the angles θ for which the y-coordinate of the point on the unit circle is $\dfrac{1}{2}$. Hence, the solutions are $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$.

(b) $3\tan\theta + 3 = 0$

Solution:
We need to find the angles θ such that $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -1$, i.e, where $\sin\theta = -\cos\theta$. So we look for the points on the unit circle where x and y are equal but of opposite sign. This occurs at $\theta = \dfrac{3\pi}{4}$ and $\theta = \dfrac{7\pi}{4}$.