Mathematics 1275/D225, Spring 2024 **Exam #2**
Instructor: Suman Ganguli **Wednesday, April 17** **Name:** ______________________

In order to receive full credit, you must **show all your work** and simplify your answers.

1. (2 points) Complete the following sentence: "The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the solutions of the equation ______________________."

Solution: The quadratic formula gives the solutions of the equation $ax^2 + bx + c = 0$.

2. (8 points) (a) Use the quadratic formula to find the solutions of the equation $x^2 - 8x + 1 = 0$:

Solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-8)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{64 - 4}}{2} = \frac{4 \pm \sqrt{60}}{2} = \frac{8 \pm 2\sqrt{15}}{2} = 4 \pm \sqrt{15}$$

(b) Now let's solve the same equation by completing the square and using the square root property.
First, we separate the constant term from the variable terms (i.e., move the constant term from the LHS to the RHS):

$$x^2 - 8x = -1$$

- Now complete the square on the LHS, i.e., compute the number that will make the LHS into a perfect square trinomial, and add it to both sides of the equation:

Solution: Given $x^2 + bx$, we add $\left(\frac{b}{2}\right)^2$ in order to complete the square. Since $b = -8$ in this instance, we add $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$ (to both sides of the equation):

$$x^2 - 8x + 16 = -1 + 16$$

- Rewrite equation by factoring the perfect square trinomial on the LHS, and simplify the RHS by adding together the two constants:

Solution: $(x - 4)^2 = 15$

- Now take the square root of both sides, remembering the square root property (i.e., there are two square roots to consider on the RHS, the positive and the negative!):

Solution: $x - 4 = \pm\sqrt{15}$

- Finally, solve for x:

Solution: $x = 4 \pm \sqrt{15}$

3. (10 points) Use the quadratic formula to solve each equation. Simplify the solutions completely.

(a) $3x^2 - 5x + 2 = 0$

Solution: Applying the quadratic formula with $a = 3, b = -5, c = 2$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(2)}}{2(3)} = \frac{5 \pm \sqrt{25 - 24}}{6} = \frac{5 \pm 1}{6}$$

so the solutions are $x = \frac{6}{6} = 1$ and $x = \frac{4}{6} = \frac{2}{3}$.

(b) $2x^2 + 8x + 10 = 0$

(Hint: this quadratic equations has two complex solutions; simplify the solutions into the form $a \pm bi$.)

Solution: Applying the quadratic formula with $a = 2, b = 8, c = 10$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - 4(2)(10)}}{2(2)} = \frac{-8 \pm \sqrt{64 - 80}}{4} = \frac{-8 \pm \sqrt{-16}}{4} = -\frac{8}{4} \pm \frac{4i}{4} = -2 \pm i$$

4. (10 points) Simplify the following complex fractions:

$$\frac{\frac{2}{x^2} + \frac{1}{x}}{\frac{4}{x^2} - \frac{1}{x}} =$$

Solution:

Since the LCD for the both the

$$\frac{\frac{2}{x^2} + \frac{1}{x}}{\frac{4}{x^2} - \frac{1}{x}} = \frac{\frac{2}{x^2} + \frac{1}{x} \cdot \frac{x}{x}}{\frac{4}{x^2} - \frac{1}{x} \cdot \frac{x}{x}} = \frac{\frac{2}{x^2} + \frac{x}{x^2}}{\frac{4}{x^2} - \frac{x}{x^2}} = \frac{\frac{2+x}{x^2}}{\frac{4-x}{x^2}} = \frac{2+x}{x^2} \cdot \frac{x^2}{4-x} = \frac{2+x}{4-x}$$

5. (10 points) Perform the indicated operations on the complex numbers. Write the result in standard complex form, i.e., in the form $a + bi$. (Remember to use the definition of i to simplify: $i^2 = -1$.)

(a)

$$(-2 - 3i)(-7 - 5i)$$

Solution:

$$(-2 - 3i)(-7 - 5i) = 14 + 10i + 21i + 15i^2 = 14 + 31i - 15 = -1 + 31i$$

(b)

$$\frac{5}{3 + i}$$

Solution:

$$\frac{5}{3+i} \times \frac{3-i}{3-i} = \frac{15-5i}{9+3i-3i-i^2} = \frac{15-5i}{9-(-1)} = \frac{15}{10} - \frac{5i}{10} = \frac{3}{2} - \frac{1}{2}i$$

(c)

$$\frac{1 + 8i}{1 - 2i}$$

Solution:

$$\frac{1+8i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{1+2i+8i+(8i)(2i)}{1+2i-2i-4i^2} = \frac{1+10i-16}{1+4} = \frac{-15+10i}{5} = \frac{-15}{5} + \frac{10i}{5} = -3 + 2i$$

Check: $(-3 + 2i)(1 - 2i) = -3 - 3(-2i) + 2i - 4i^2 = -3 + 6i + 2i + 4 = 1 + 8i$✓

6. (10 points) Algebraically find the vertex, y-intercept, and x-intercept(s) for the graph of the quadratic function

$$y = x^2 + 2x - 3$$

and then sketch the graph.

(a) y-intercept:
(Hint: the y-intercept corresponds to the value of y when $x = 0$; for a quadratic function $y = ax^2 + bx + c$, clearly $y = c$ when $x = 0$, and so the y-intercept of the graph is $(0, c)$).

Solution:
$y = -3$ when $x = 0$ so the y-intercept is $(0, -3)$

(b) x-intercepts:
(Hint: find the solutions of the quadratic equation $x^2 + 2x - 3 = 0$ by factoring and using the zero product property. Alternatively, you can use the quadratic equation.)

Solution:
Since $x^2 + 2x - 3 = (x + 3)(x - 1)$, the solutions of $x^2 + 2x - 3 = 0$ occur for $x + 3 = 0$ and $x - 1 = 0$, i.e., $x = -3$ and $x = 1$.
Thus, the x-intercepts of the graph are the points $(-3, 0)$ and $(1, 0)$.

(c) You can find the vertex of $y = x^2 + 2x - 3$ by either of two methods:
(i) put the function in the vertex form

$$y = (x - h)^2 + k$$

by completing the square (in which case the vertex is at (h, k)); or
(ii) use the vertex formula, which says that the x-coordinate of the vertex of $y = ax^2 + bx + c$ is

$$x = -\frac{b}{2a}$$

(Extra credit: find the vertex by both methods!)

Solution:
(i) completing the square:
$y + 3 = x^2 + 2x$
$y + 3 + 1 = x^2 + 2x + 1$
$y = (x + 1)^2 - 4$
So the vertex is at $(-1, -4)$.
(ii) Since $a = 1$ and $b = 2$ for this quadratic function, the x-coordinate of the vertex is

$$x = -\frac{2}{2(1)} = -1$$

and we get the x-coordinate by plugging $x = -1$ into $y = x^2 + 2x - 3$:

$$y = (-1)^2 + 2(-1) - 3 = 1 - 2 - 3 = -4$$

So again we find that the vertex is at $(-1, -4)$.

(d) Sketch the graph of $y = x^2 + 2x - 3$. Label the vertex, y-intercept, and x-intercepts of the parabola with their coordinates: