MAT 1475 Final Exam Review Problems

Revised by Prof. Kostadinov, Prof Africk and Prof Colucci: Fall 2010 through Fall 2015
Revised by Prof. Africk , Colucci,Douglas, Johnstone, Rozenblyum, Taraporevala: Spring 2023

#1 Evaluate the following Limits:

a) $\lim\limits_{x\to 0} \dfrac{e^{6x}-1}{\sin 3x}$ b) $\lim\limits_{x\to\infty}\left(\left(\dfrac{2}{3}\right)^x + \dfrac{7x-5x^2}{x^3-x^2}+6\right)$ c) $\lim\limits_{x\to 0}\dfrac{2x-1+\cos 4x}{1-e^{4x}}$ d) $\lim\limits_{x\to\infty}\left(\left(\dfrac{4}{5}\right)^x+\dfrac{6x^2-3}{6x+2x^2}+5\right)$

#2 Find the derivatives of the following functions using the Limit definition of derivative:

a) $f(x)=2x^2-5x$ b) $f(x)=-2x^2+3x-4$

#3 Find the derivative $y'=\dfrac{dy}{dx}$ of the following functions, using the derivative rules:

a) $y=\cos^4 6x$ b) $y=x\cdot\sin 4x$ c) $y=x\cdot\cos 3x$ d) $y=x\cdot e^{2x}$ e) $y=x\cdot e^{-x}$ f) $y=\sin^{-1}(x^2)$ g) $y=\tan^{-1}(x^3)$

#4 Find the derivatives of the following functions:

a) $f(x)=3x^4\sec(5x)$ b) $f(x)=\sqrt{x}\tan(3x)$ c) $f(x)=\dfrac{4x^2-5}{2x^2-1}$ d) $f(x)=e^{3x}\cos(5x)$ e) $f(x)=\dfrac{2x^2}{x^2-16}$

#5 Find the derivative $y'=\dfrac{dy}{dx}$ of the following functions:

a) $y=x^x$ b) $y=(\sin x)^{7x}$ c) $y=(\sqrt{x})^{\cos x}$

#6 Find the equation of the tangent line, in slope-intercept form, to the following curves:

a) $f(x)=2x^3+5x^2+6$ at $(-1,9)$ b) $f(x)=4x-x^2$ at $(1,3)$

#7 Using implicit differentiation, find the equation of the tangent line to the given curve at the given point:

a) $3x^2y^2-3y-17=5x+14$ at $(1,-3)$ b) $y^2-7xy+x^3-2x=9$ at $(0,3)$

#8 (a) Find the equation of the tangent line to $y=\sqrt{x+3}$ at $x=6$

(b) Find the differential dy at $y=\sqrt{x+3}$ and evaluate it for $x=6$ and $dx=0.3$

#9 Wire of length 20m is divided into two pieces and the pieces are bent into a square and a circle. How should this be done in order to minimize the sum of their areas? Round your answer to the nearest hundredth.

#10 Find the dimensions of a closed box having a square base with surface area 12 and maximal volume.

#11 If a snowball melts so its surface area decreases at a rate of $1\,\text{cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 6 cm.

#12 The radius of a sphere increases at a rate of 3 in/sec. How fast is the volume increasing when the diameter is 24in?

#13 The radius of a cone is increasing at a rate of 3 inches/sec, and the height of the cone is 3 times the radius. Find the rate of change for the volume of that cone when the radius is 7 inches.

#14 Sand pours from a chute and forms a conical pile whose height is always equal to its base diameter. The height of the pile increases at a rate of 5 feet/hour. Find the rate of change of the volume of the sand in the conical pile, when the height of the pile is 4 feet.

#15 A cylindrical tank with radius 8 m is being filled with water at a rate of 2 m^3/min. What is the rate of change of the water height in this tank?

#16 A box with a square base and an open top must have a volume of 256 cubic inches. Find the dimensions of the box that will minimize the amount of material used (the surface area).

#17 A farmer wishes to enclose a rectangular plot using 200 m of fencing material. One side of the land borders a river and does not need fencing. What is the largest area that can be enclosed?

#18 For the function $y = x^3 - 3x^2 - 1$, use derivatives to:

(a) determine the intervals of increase and decrease.

(b) determine the local (relative) maxima and minima.

(c) determine the intervals of concavity.

(d) determine the points of inflection.

(e) sketch the graph with the above information indicated on the graph.

#19 Evaluate each the following definite integrals: a) $\int_{-2}^{2}(4e^x + 3)dx$ b) $\int_{1}^{4}(2e^x - 2)dx$

#20 Evaluate each the following indefinite integrals: a) $\int \frac{5x^7 - 2x^4}{x^2}dx$ b) $\int \frac{3x^6 + 2x^2}{x^4}dx$

Answers to questions:

#1 a) 2 b) 6 c) $-\frac{1}{2}$ d) 8

#2 a) $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2 - 5(x+h) - (2x^2 - 5x)}{h} = \lim_{h\to 0}\frac{4xh + 2h^2 - 5h}{h} = \lim_{h\to 0}(4x + 2h - 5) = 4x - 5$

b) $\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{-2(x+h)^2 + 3(x+h) - 4 - (-2x^2 + 3x - 4)}{h} = \lim_{h\to 0}\frac{-4xh - 2h^2 + 3h}{h} = \lim_{h\to 0}(-4x - 2h + 3) = -4x + 3$

#3 a) $-24\cos^3 6x \sin 6x$ b) $\sin 4x + 4x\cos 4x$ c) $\cos 3x - 3x\sin 3x$ d) $e^{2x} + 2xe^{2x}$ e) $e^{-x} - xe^{-x}$

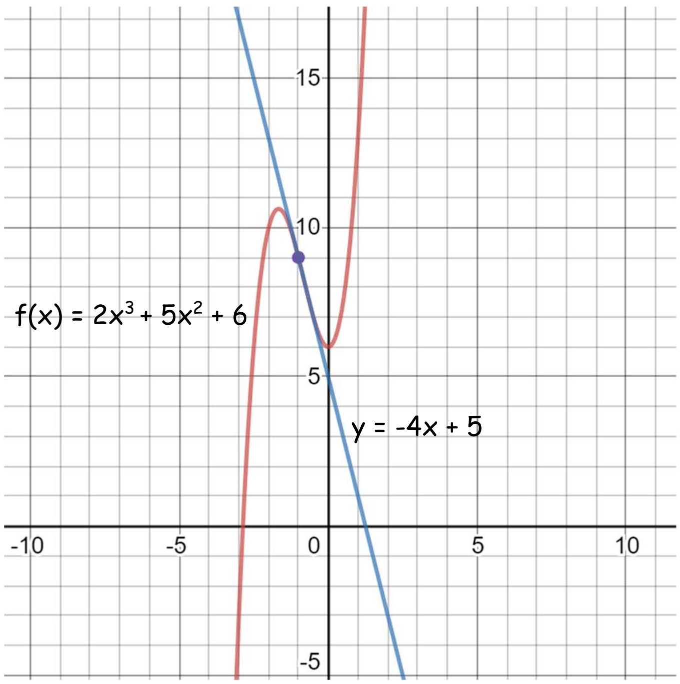
f) $\frac{2x}{\sqrt{1-x^4}}$ g) $\frac{3x^2}{x^6+1}$

#4 a) $12x^3\sec(5x) + 15x^4\sec(5x)\tan(5x)$ b) $\frac{1}{2}\frac{\tan(3x)}{\sqrt{x}} + 3\sqrt{x}\left(1 + \tan^2(3x)\right)$ c) $\frac{12x}{(2x^2-1)^2}$

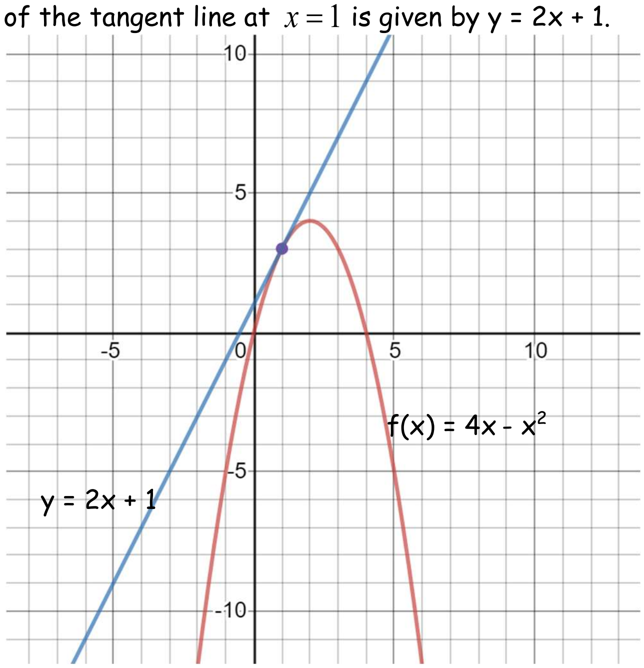
d) $3e^{3x}\cos 5x - 5e^{3x}\sin 5x$ e) $\frac{-64x}{(x^2-16)^2}$

#5 a) $y' = (1+\ln x)x^x$ b) $y' = (\sin x)^{7x}(7x\cot x + 7\ln\sin x)$ c) $\frac{1}{2}\sqrt{x}^{\cos x}\left(\frac{1}{x}\cos x - \sin x \ln x\right)$

#6 a) The equation of the tangent line at $x=-1$ is given by y = -4x + 5.

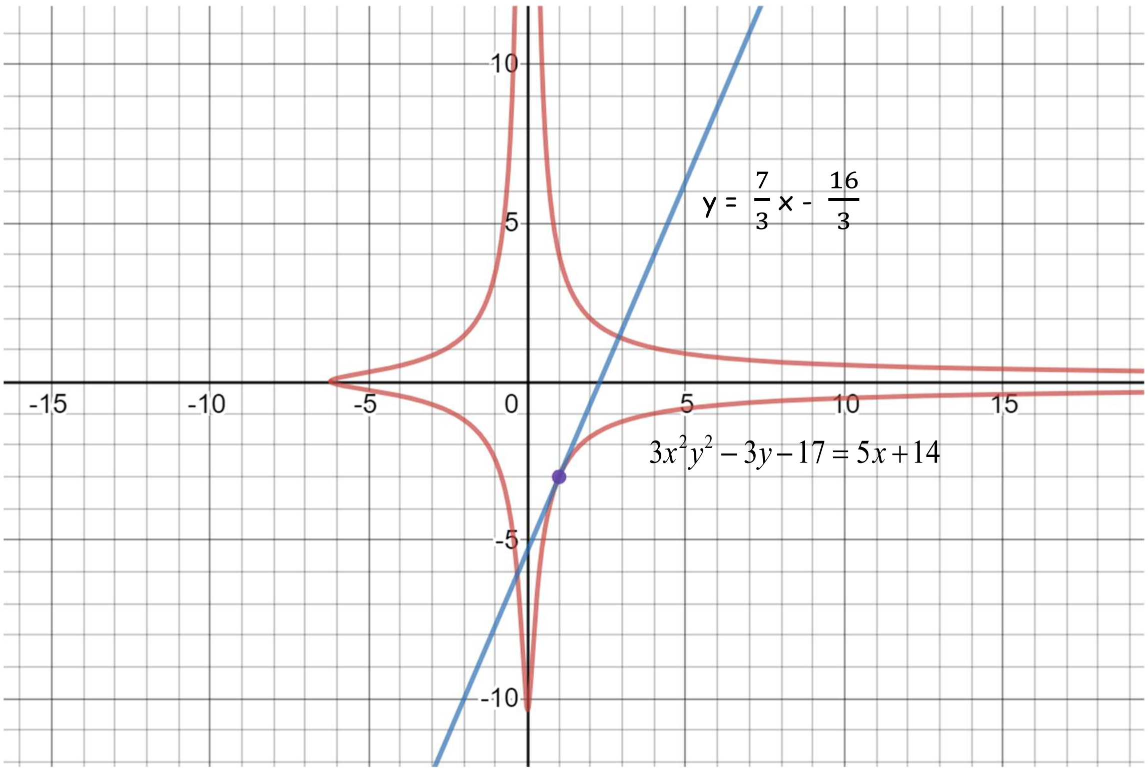


b) The equation of the tangent line at $x=1$ is given by y = 2x + 1.

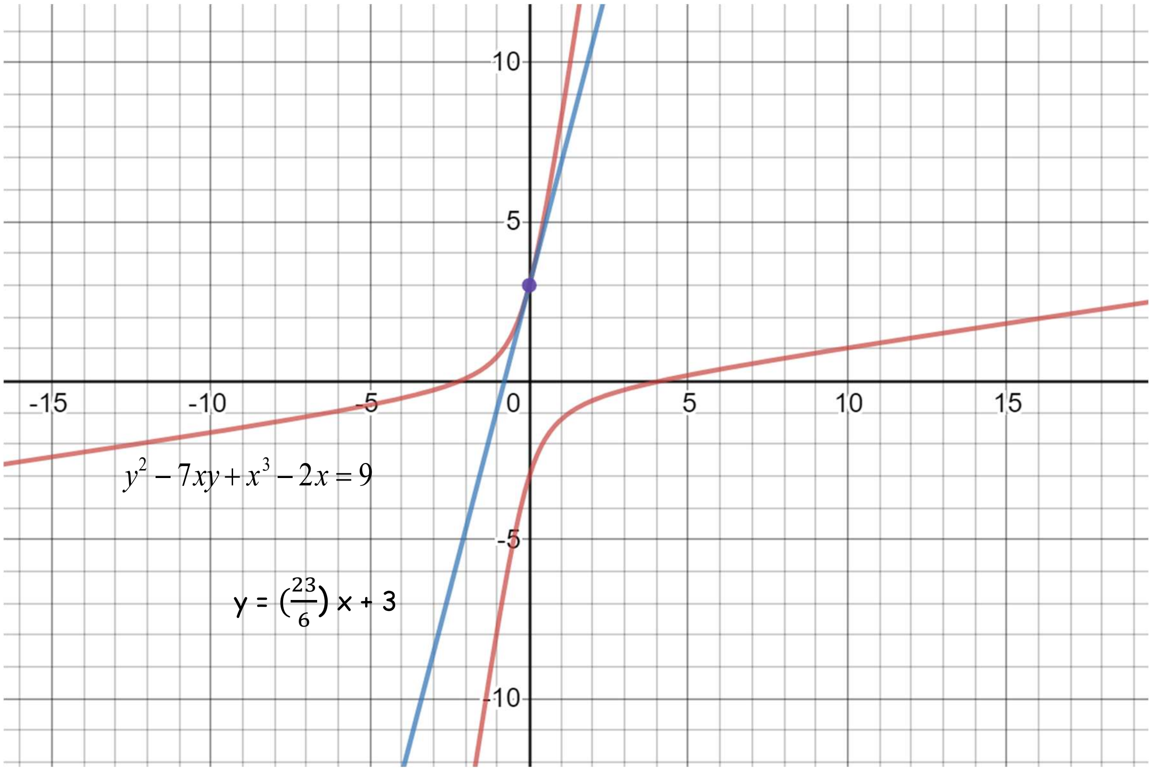


#7 a) The derivative $y' = \dfrac{5-6xy^2}{6x^2y-3}$ one computes by implicit differentiation. The slope of the tangent line at the given point is the derivative evaluated at (1,-3), that is $y'(1) = 7/3$. The equation of the tangent line is given by:

$y = -3 + \frac{7}{3}(x - 1) = \frac{7}{3}x - \frac{16}{3}$



#7 b) y' = $\frac{7y-3x^2+2}{2y-7x}$, y'(0) = $\frac{23}{6}$, y = 3 + $\left(\frac{23}{6}\right)$ (x - 0) = $\left(\frac{23}{6}\right)$ x + 3



#8 a) y = $\frac{1}{6}$ x + 2 b) dy = $\frac{1}{2\sqrt{x+3}}$ dx , dy = .05 when x = 6 and dx = 0.3

#9 Let the first piece have a length x, then the second one has a length 20-x. From the first piece, we can form a square with a side of length $\frac{x}{4}$ and an area $\left(\frac{x}{4}\right)^2$. From the second piece, we can form a circle with circumference $20 - x = 2\pi r$ thus having a radius $r = \frac{20-x}{2\pi}$ and an area $\pi r^2 = \pi\left(\frac{20-x}{2\pi}\right)^2$. We want to minimize the sum of the two areas : $f(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{20-x}{2\pi}\right)^2$. This is done by setting the derivative to zero $f'(x) = \frac{(4+\pi)x - 80}{8\pi} = 0$ and solving for the critical point $x_0 = \frac{80}{4+\pi} \approx 11.2$ at which the (global) minimum of f(x) is attained (why?), and which gives the answer how the wire should be divided in order to minimize the sum of the two areas.

#10 Let the length of one side of the base square be x and the height of the box be y. The total surface area of the box is $4xy + 2x^2 = 12$ and the volume is $V = x^2 y$. Express y from the first equation $y = \frac{6-x^2}{2x}$ and plug it into the volume: $V = \frac{x(6-x^2)}{2}$. Set the derivative to zero to find the critical point(s): $V'(x) = 3 - \frac{3}{2}x^2 = 0 \Rightarrow x = \pm\sqrt{2}$ and take $x = \sqrt{2}$ since length is positive. The volume attains a (local) maximum at $x = \sqrt{2}$, which is also a global maximum for $x > 0$ (why?).

#11 The rate of change of the diameter is $\frac{dD}{dt} = -\frac{1}{12\pi} \approx -0.0265\ cm/\min$

#12 The rate of change of the volume is $\frac{dV}{dt} = 1728\pi$ in^3/sec

#13 $\frac{dV}{dt} = 441\pi \approx 1385.4\ in^3/\sec$ (Note: $V = \frac{1}{3}\pi r^2 h = \pi r^3$, since $h = 3r$)

#14 $\frac{dV}{dt} = 20\pi \approx 62.8\ ft^3/hour$ (Note: $V = \frac{1}{12}\pi h^3$, since $r = \frac{h}{2}$)

#15 $V = \pi r^2 h = 64\pi h \rightarrow \frac{dV}{dt} = 64\pi \frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{1}{64\pi}\frac{dV}{dt} = \frac{2}{64\pi} = \frac{1}{32\pi}$ m/min

#16 The base is 8 and height is 4, thus the dimensions are: $8 \times 8 \times 4$

#17 The area is maximized when one side of the rectangle is 50m and the other is 100m, which gives an area of 5000m^2.

Note: For problems #18, one may use a graphing calculator to plot the functions and confirm the analysis.
Please note that students are allowed to use graphing calculators on the final exam in MAT 1475

#18 The critical points are $x = 0, 2$ (where $y' = 3x(x-2) = 0$). The derivative $y' > 0$ to the left of 0 and to the right of 2, thus the function is increasing there; and $y' < 0$ between 0 and 2, thus the function is decreasing there. The first derivative test tells us that the function has a local (relative) maximum at the point $(0,-1)$ (as the function is increasing to the left of 0 and decreasing to the right of 0), as well as a local (relative) minimum at the point $(2,-5)$ (as the function is decreasing to the left of 2 and increasing to the right of 2). The second derivative $y'' = 6x - 6$ becomes zero at $x = 1$; it is negative to the left of 1, thus the function is concave downward there, and positive to the right of 1, so the function is concave upward there. Since the second derivative changes sign at $x = 1$, the point on the curve $(1,-3)$ is the only inflection point.

Answers:

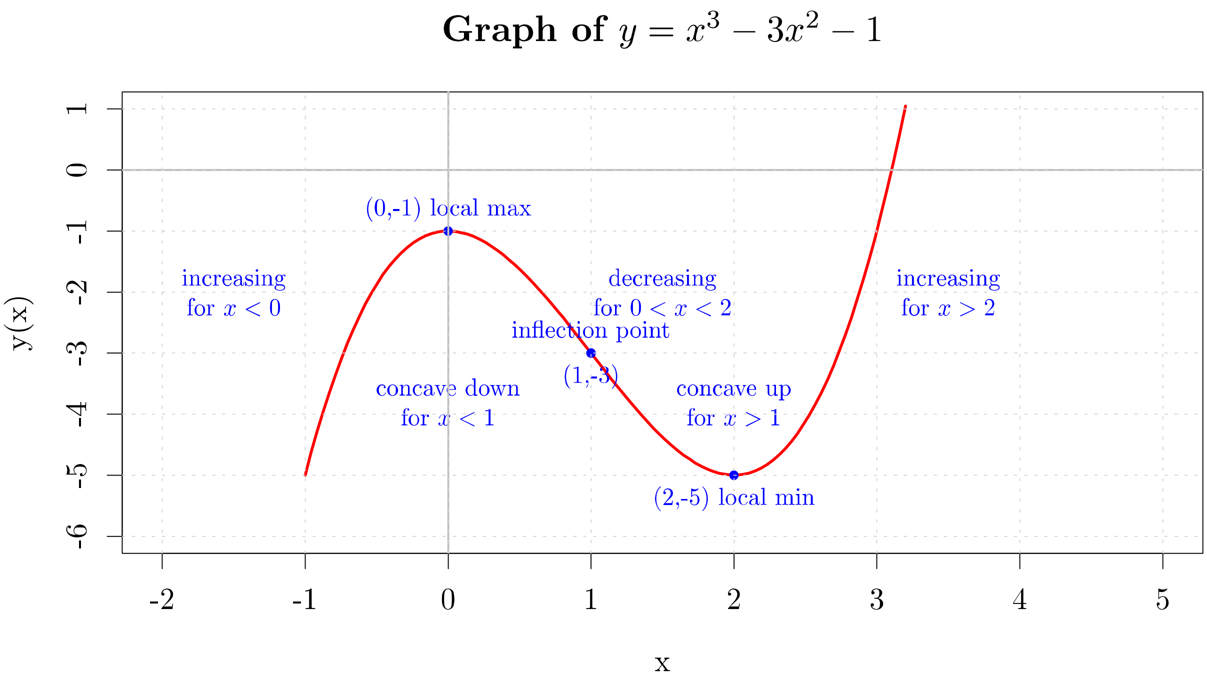
(a) The function is increasing for x<0 and 2<x and decreasing for 0<x<2.

(b) The function has a local maximum at (0,-1) and a local minimum at (2,-5).

(c) The function is concave down for x<1 and concave up for 1<x.

(d) The function has an inflection point at (1,-3)

(e) See the graph below.



#19 a) $\int_{-2}^{2}(4e^x+3)dx = 4\int_{-2}^{2} e^x\,dx + \int_{-2}^{2} 3\,dx = 4(e^2-e^{-2})+3(2-(-2)) = 4(e^2-e^{-2})+12 \approx 41.015$

b) $\int_{1}^{4}(2e^x-2)dx = 2\int_{1}^{4} e^x\,dx - 2\int_{1}^{4} dx = 2(e^4-e^1)-2(4-1) = 2(e^4-e)-6 \approx 97.76$

#20 a) $\int \frac{5x^7-2x^4}{x^2}dx = \int (5x^5-2x^2)dx = 5\int x^5\,dx - 2\int x^2\,dx = \frac{5x^6}{6}-\frac{2x^3}{3}+C$

b) $\int \frac{3x^6+2x^2}{x^4}dx = 3\int x^2\,dx + 2\int x^{-2}\,dx = \frac{3x^3}{3}+\frac{2x^{-2+1}}{(-2+1)}+C = x^3-2x^{-1}+C = x^3-\frac{2}{x}+C$