MAT1475, Fall 2023 **Exam #1**
Instructor: Suman Ganguli **Wednesday, October 18** **Name:** ____________________

Question:	1	2	3	4	5	Total
Points:	10	10	10	15	5	50
Score:						

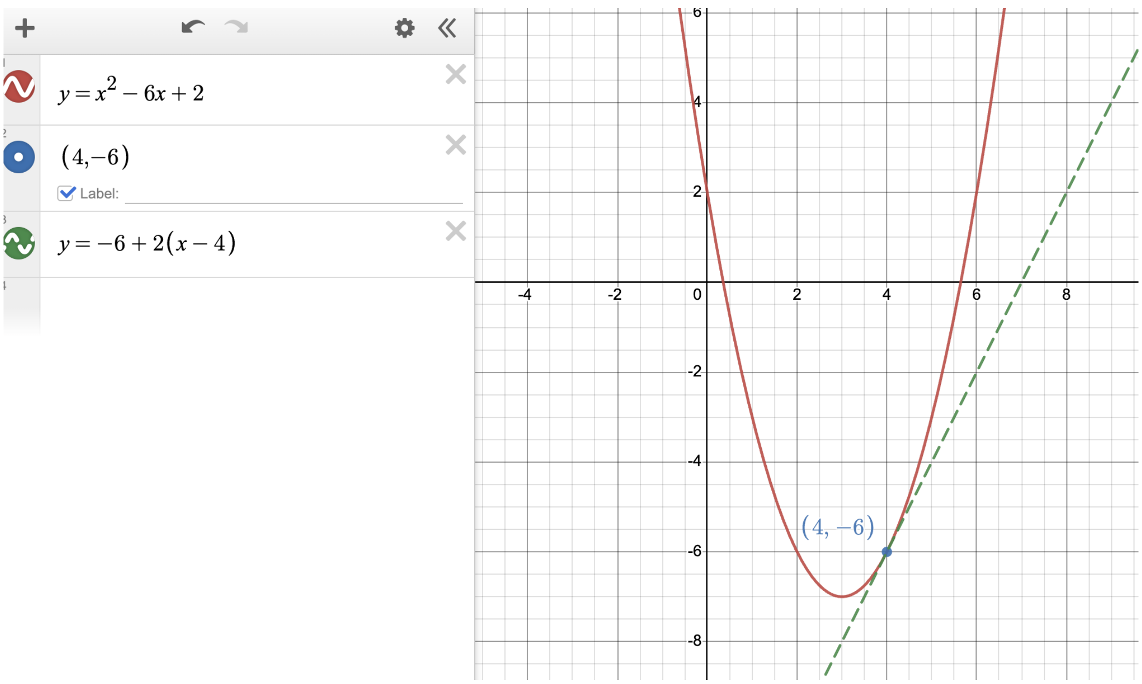
1. (10 points) Consider the quadratic function $f(x) = x^2 - 6x + 2$.

   (a) Calculate $f(4)$:

   Solution:

   $$f(4) = 4^2 - 6(4) + 2 = 16 - 24 + 2 = -6$$

   (b) Shown is the graph of $y = x^2 - 6x + 2$. Plot the point $(4, f(4))$ and sketch the tangent line at that point:



   (c) Calculate $f'(4)$ using the limit definition of the derivative:

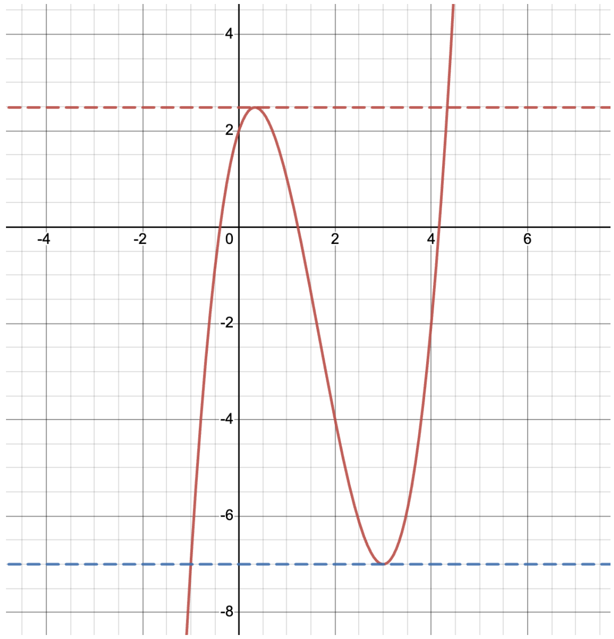
   Solution:

   $$f'(4) = \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4} \frac{x^2 - 6x + 2 - (-6)}{x - 4} = \lim_{x\to 4} \frac{x^2 - 6x + 8}{x - 4} = \lim_{x\to 4} \frac{(x-4)(x-2)}{x - 2} = \lim_{x\to 4}(x-2) = 4-2 = 2$$

   (d) Now find $f'(x)$ using the differentiation rules, and use that to calculate $f'(4)$.
   (You should get the same value as in part (c).)

   Solution: $f'(x) = 2x - 6 \Longrightarrow f'(4) = 2(4) - 6 = 2$

2. (10 points) Shown below is the graph of $g(x) = x^3 - 5x^2 + 3x + 2$



(a) The graph has two points where the tangent line is horizontal, i.e., the tangent line has slope zero. Circle those two points on the graph and sketch the horizontal tangent lines.

(b) Now find the x-coordinates of those two points:

- Find the derivative of $g(x)$:

$$g'(x) = \frac{d}{dx}(x^3 - 5x^2 + 3x + 2) =$$

Solution: $g'(x) = 3x^2 - 10x + 3$

- Recall that the quadratic formula gives the two solutions of a quadratic equation $ax^2 + bx + c = 0$ as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use the quadratic formula to find the two solutions of the equation $g'(x) = 0$:

Solution: The solutions of

$$g'(x) = 3x^2 - 10x + 3 = 0$$

are:

$$x = \frac{10 \pm \sqrt{10^2 - 4(3)(3)}}{2(3)} = \frac{10 \pm \sqrt{64}}{6} = \frac{10 \pm 8}{6}$$

Thus, the two points where the tangent line to the graph is horizontal occur at $x = 18/6 = 3$ and $x = 2/6 = 1/3$.

3. (10 points) Consider the function $y = -\frac{1}{x} = -x^{-1}$

(a) Find the derivative of this function. (Hint: use the Power Rule.)

$$\frac{dy}{dx} =$$

Solution: $\frac{dy}{dx} = -(-x^{-2}) = \frac{1}{x^2}$

(b) Calculate the value of $\frac{dy}{dx}$ at $x = 2$:

$$\left.\frac{dy}{dx}\right|_{x=2} =$$

Solution: $\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2^2} = \frac{1}{4}$

(c) Write the equation of the tangent line to $y = -\frac{1}{x}$ at the point $\left(2, -\frac{1}{2}\right)$. (Use the point-slope equation of a line.)

Solution:

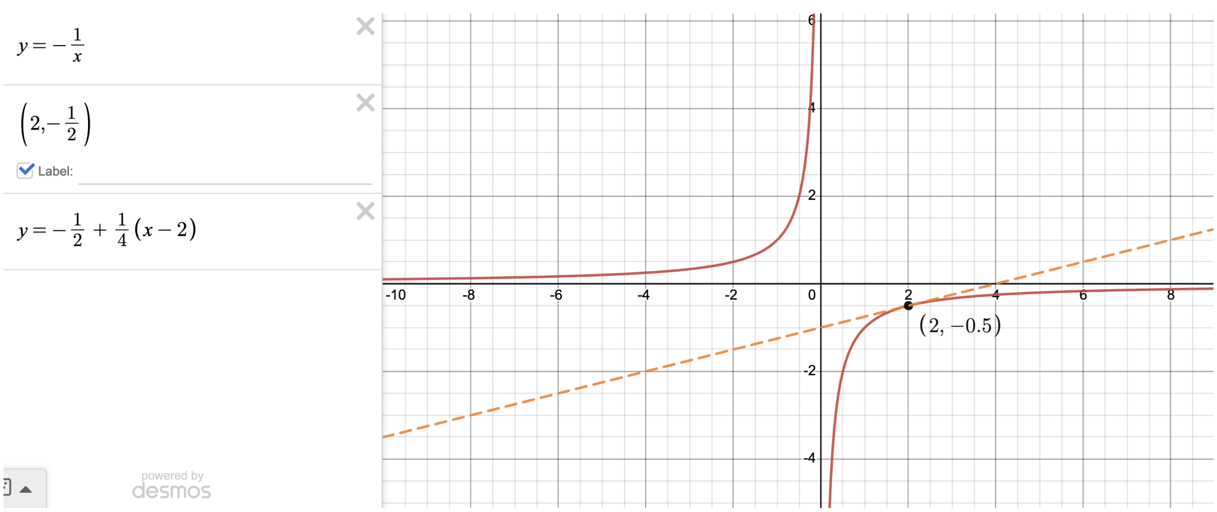
$$y = -\frac{1}{2} + \frac{1}{4}(x - 2)$$

This can be transformed to slope-intercept form:

$$y = -\frac{1}{2} + \frac{1}{4}x - \frac{1}{4}$$

$$y = \frac{1}{4}x - \frac{3}{4}$$

(d) Shown below is the graph of $y = -\frac{1}{x}$. Label the point $\left(2, -\frac{1}{2}\right)$ and sketch the tangent line.



4. (15 points) Find the derivatives of the following functions, using the differentiation rules:

   (a) $h(t) = 100 + 45t - 5t^2$

   Solution:

   $$\frac{dh}{dt} = 45 - 10t$$

   (b) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

   Solution:

   $$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}} =$$

   (c) $s(t) = (t^4 + 6t + 1)(2t^3 + t^2 + t + 1)$
   Apply the Product Rule – you don't have to simplify your solution:

   Solution:

   $$\frac{ds}{dt} = (4t^3 + 6)(2t^3 + t^2 + t + 1) + (t^4 + 6t + 1)(6t^2 + 2t + 1)$$

   (d) $y = \frac{x^2}{1 - x^2}$
   Apply the Quotient Rule – please attempt to simplify the numerator in your solution:

   Solution:

   $$\frac{dy}{dx} = \frac{(2x)(1 - x^2) - (x^2)(-2x)}{(1 - x^2)^2} = \frac{(2x - 2x^3) - (-2x^3)}{(1 - x^2)^2} = \frac{2x}{(1 - x^2)^2}$$

5. (5 points) Explain (in 2-3 sentences) the meaning and definition of m_{sec} in terms of the graph shown. Also explain how m_{sec} is used in the limit definition of the derivative $f'(a) = m_{tan}$.

   (Hint: Use terms and concepts such as secant line, slope, rise/run, difference quotient, tangent line, and limits, and use the notation for the points shown on the graph. You can also annotate the graph as part of your explanation.)

   Solution: Here is a sample explanation:

   m_{sec} is defined as the quotient of the "rise" between the two points $(f(a+h) - f(a))$ over the "run" $(a+h) - a = h$. Thus, m_{sec} is the slope of the secant line shown on the graph, i.e., the line passing through the two points on the curve, $(a, f(a))$ and $(a+h, f(a+h))$. This is what we called the difference quotient, $\frac{\Delta f}{\Delta x}$.

   The derivative $f'(a)$ is defined as the limit of this difference quotient as h goes to zero:

   $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   This is because as h goes to zero, the point $(a+h, f(a+h))$ gets closer and closer to $(a, f(a))$, and (the slope of) the secant line m_{sec} converges to (the slope of) the tangent line, i.e., $m_{tan} = f'(a)$.