MAT1575, Spring 2023 **Exam #3 (take-home)**
Instructor: Suman Ganguli **Due: Mon Dec 18, 2023** **Name:** ______________________

Question:	1	2	3	4	5	6	Total
Points:	5	5	5	5	5	5	30
Score:							

In order to receive full credit, you must **show all your work**, and write out your solutions in a clear and organized manner. **Please work on this exam individually.** You can (and should) consult resources such as your class notes, the textbook, the Final Exam Review solutions, etc. in order to work through these exercises.

1. (5 points) For the following geometric series

- state the values of a and r
- determine whether the series converges or diverges, based on the value of r
- if the series converges, compute what value it converges to

(a)

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n}$$

Solution: Diverges as geometric series with $r = 3/2 > 1$

(b)

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n}$$

Solution: Converges as geometric series with $r = 2/3 < 1$. The initial term is $a = 2/3$. Hence, the series converges to

$$\frac{a}{1-r} = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = 2$$

For #2-7, determine whether the infinite series converges or diverges. Justify your answer by using an appropriate test:

2. (5 points)

$$\sum_{n=1}^{\infty} \frac{7n^4}{10n^4 + n^2 + 1}$$

Solution: Diverges by the Divergence Test, since:

$$\lim_{n\to\infty} \frac{7n^4}{10n^4 + n^2 + 1} = \lim_{n\to\infty} \frac{7n^4}{n^4(10 + 1/n^2 + 1/n^4)} = \frac{7}{10} \neq 0$$

3. (5 points) Use the p-series test for the following. State the value of p in each case.

(a)

$$\sum_{n=1}^{\infty} n^{-0.05}$$

Solution: Diverges since it's a p-series with $p = 0.05 < 1$:

$$\sum_{n=1}^{\infty} n^{-0.05} = \sum_{n=1}^{\infty} \frac{1}{n^{0.05}}$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.05}}$$

Solution: Converges since it's a p-series with $p = 1.05 > 1$

4. (5 points) Use the limit-comparison test for the following:

(a)

$$\sum_{n=0}^{\infty} \frac{1}{9n^2+10}$$

Solution: Converges by Limit Comparison with $\sum \frac{1}{n^2}$ (which we know converges as a p-series with $p = 2 > 1$):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{9n^2+10}\frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2}{n^2(9+10/n^2)}\frac{n}{1} = \frac{1}{9}$$

(b)

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{9n^2+10}}$$

Solution: Diverges by Limit Comparison with $\sum \frac{1}{n}$ (which we know diverges - the harmonic series):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{9n^2+10}}\frac{n}{1} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2(9+10/n^2)}}\frac{n}{1} = \lim_{n\to\infty} \frac{1}{n\sqrt{(9+10/n^2)}}\frac{n}{1} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

5. (5 points) Use the Ratio Test:

$$\sum_{n=1}^{\infty} \frac{n^2}{7^n}$$

Solution: Converges by Ratio Test since $\rho = \frac{1}{7} < 1$:

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{(n+1)^2}{7^{n+1}}\frac{7^n}{n^2} = \lim_{n\to\infty} \frac{1}{7} \cdot \frac{n^2+2n+1}{n^2} = \frac{1}{7}$$

6. (5 points) Recall that an alternating series may be absolutely convergent, conditionally convergent, or divergent.

(a) Explain why the following alternating series is absolutely convergent:

$$\sum_{n=1}^{\infty}(-1)^n\, n^{-5}$$

Solution: Absolutely convergent since $\sum \frac{1}{n^5}$ converges as p-series with $p = 5 > 1$

(b) Show that the following alternating series is conditionally convergent:

$$\sum_{n=0}^{\infty}(-1)^n \frac{n^3+1}{n^4+1}$$

(Hint: You can use the fact that $\frac{1^3+1}{1^4+1} > \frac{2^3+1}{2^4+1} > \frac{3^3+1}{3^4+1} > \dots$)

Solution: This alternating series is *not* absolutely convergent since $\sum \frac{n^3+1}{n^4+1}$ diverges by Limit Comparison with $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+1}{n^4+1}\frac{n}{1} = \lim_{n\to\infty} \frac{n^4(1+1/n^3)}{n^4(1+1/n^4)} = 1$$

On the other hand, $\sum_{n=0}^{\infty}(-1)^n \frac{n^3+1}{n^4+1}$ *is* convergent, by the Alternating Series Test:

- as given in the hint, $\frac{1^3+1}{1^4+1} > \frac{2^3+1}{2^4+1} > \frac{3^3+1}{3^4+1} > \dots$
- and $\lim_{n\to\infty} \frac{n^3+1}{n^4+1} = 0$

Hence, the alternating series is *conditionally* convergent.