Convergence/Divergence of Infinite Series

Math 1575 - Fall 2023

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Section numbers refer to *OpenStax Calculus, Volume 2* by E. Herman and G. Strang:

Section 5.2

Geometric Series

If the series is geometric, you can determine convergence or divergence based on the value of r:

- A geometric series *converges* if $|r| < 1$ (i.e., if $-1 < r < 1$), in which case

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$$

$$\sum_{n=M}^{\infty} ar^n = ar^M + ar^{M+1} + ar^{M+2} + ar^{M+3} + \ldots = \frac{ar^M}{1-r}$$

- A geometric series *diverges* if $|r| \geq 1$ (i.e., $r \leq -1$ or $r \geq 1$).

Examples:

- OpenStax Example 5.9
- Final Exam Review, #8(b)

Section 5.3

Divergence (or nth-Term) Test

If the individual terms in the series don't go to zero, then the series diverges:

- An infinite series $\sum a_n$ *diverges* if the nth term a_n does not go to zero, i.e., if

$$\lim_{n\to\infty} a_n \neq 0$$

- But the "converse" is not true, i.e., we have numerous examples of infinite series $\sum a_n$ where $\lim_{n\to\infty} a_n = 0$ but $\sum a_n$ diverges.

Examples:

- OpenStax Example 5.13
- Final Exam Review, #8(a)

Integral Test

If you can integrate the function that makes up the terms in the series, you can determine convergence or divergence based on the improper integral:

- Suppose $a_n = f(n)$, where $f(x)$ is positive, decreasing, and continuous for $x \geq M$.

  (i) If the improper integral $\displaystyle\int_M^\infty f(x)\,dx$ converges, then the series $\displaystyle\sum_{n=M}^\infty a_n$ also converges.

  (ii) If the improper integral $\displaystyle\int_M^\infty f(x)\,dx$ diverges, then the series $\displaystyle\sum_{n=M}^\infty a_n$ also diverges.

Examples:

- OpenStax Example 5.14

p-series Test

By applying the Integral Test to $\displaystyle\sum_{n=M}^\infty \frac{1}{n^p}$, i.e., by integrating $\displaystyle\int_M^\infty \frac{1}{x^p}\,dx$, we showed that we can determine the convergence or divergence of a p-series $\displaystyle\sum_{n=M}^\infty \frac{1}{n^p}$ based on the value of p:

- If $p > 1$, then the series $\displaystyle\sum_{n=M}^\infty \frac{1}{n^p}$ converges.

- If $p \leq 1$, then the series $\displaystyle\sum_{n=M}^\infty \frac{1}{n^p}$ diverges.

Examples:

- OpenStax Example 5.15

Section 5.4

Limit Comparison Test

To test the convergence of an infinite series $\sum a_n$, you can sometimes compare it to another series $\sum b_n$ (where you know about the convergence of the latter series) by looking at the limit of a_n over b_n as n goes to infinity:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If $L > 0$, i.e., the limit is some finite number greater than 0, then $\sum a_n$ has the same convergence/divergence behavior as $\sum b_n$, i.e.,

(i) If $\sum b_n$ converges, then $\sum a_n$ also converges

(ii) If $\sum b_n$ diverges, then $\sum a_n$ also diverges

(There are additional parts of the Limit Comparison Test given in the text, but focus on this case.)

When does the Limit Comparison Test work on a given $\sum a_n$, and what's the strategy for choosing the series $\sum b_n$?

- Start by thinking about what happens to the terms a_n as n gets big. In many instances, $a_n \approx b_n$, where we already know about the convergence of $\sum b_n$ by some other method.
- Many applications of the Limit Comparison Test occur when a_n is a ratio involving polynomials and/or roots of polynomials. In such cases, a choice of $b_n = \frac{1}{n^p}$ for a certain p-value will often work.
- How do you figure out what value of p? Analyze what happens to a_n as n gets big by looking at *the leading terms* in the polynomials involved.

Example:

- Given $\displaystyle\sum_{n=1}^{\infty} \frac{12n+5}{7n^5 - n^2 + 10}$, look at the leading terms to analyze what happens as n gets big:

$$a_n = \frac{12n+5}{7n^5 - n^2 + 10} \approx \frac{12n}{7n^5} = \frac{12}{7n^4}$$

This indicates that we should use a Limit Comparison Test with $b_n = \frac{1}{n^4}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{12n+5}{7n^5 - n^2 + 10}\,\frac{n^4}{1} = \lim_{n\to\infty} \frac{n(12+\frac{5}{n})}{n^5(7 - \frac{1}{n^3} + \frac{10}{n^5})}\,\frac{n^4}{1} = \lim_{n\to\infty} \frac{12+\frac{5}{n}}{7 - \frac{1}{n^3} + \frac{10}{n^5}} = \frac{12}{7}$$

So $L = \frac{12}{7} > 0$ and we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges as a p-series with $p > 1$. Hence part (ii) of the Limit-Comparison Theorem above applies, and so $\displaystyle\sum_{n=1}^{\infty} \frac{12n+5}{7n^5 - n^2 + 10}$ also converges.

Examples:

- OpenStax Example 5.18
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Section 5.4

Absolute & Conditional Convergence of Alternating Series

Absolute Convergence: An infinite series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges (i.e., the series converges if you make all the terms positive).
Conditional Convergence: An infinite series $\sum a_n$ *converges conditionally* if $\sum a_n$ converges but it does not converge absolutely, i.e., $\sum |a_n|$ diverges.

Absolute Convergence Implies Convergence

One way of checking whether an alternating series converges is to check whether the series when you make all the terms positive converges. If so, the alternating series also converges:

- Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ also converges.
- Example: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ converges absolutely for any $p > 1$ (since the p-series test tells us that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$). Hence $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ converges for any $p > 1$.

Alternating Series Test (Theorem 2, p570)

In general it is easier to establish that an alternating series converges–you just need to check that the individual terms (without the alternating signs) are decreasing and that they go to zero:

- Alternating Series Test: An alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if
  (i) $a_1 > a_2 > a_3 > \dots$
  (ii) $\lim_{n\to\infty} a_n = 0$
- Example: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges by the Alternating Series Test (since $1 > \frac{1}{2} > \frac{1}{3} > \dots$ and $\lim_{n\to\infty}\frac{1}{n} = 0$) but $\sum_{n=1}^{\infty}|(-1)^{n+1}\frac{1}{n}| = \sum \frac{1}{n}$ diverges (the harmonic series!). Hence, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges conditionally. In fact, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ converges conditionally for any $0 < p \leq 1$.

Examples:

- OpenStax Example 5.18
- Final Exam Review, #9 (which also use the geometric series and limit-comparison, p-series, and divergence tests!)

Section 5.6: Ratio Test

Another way to test the convergence of an infinite series $\sum a_n$ is to look at the limit of the ratio of successive terms as n goes to infinity:

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

- If $\rho < 1$, then the series converges absolutely.
- If $\rho > 1$, then the series diverges.
- If $\rho = 1$, then the test is inconclusive.

The Ratio Test often works for series where a_n involves n as an exponent and/or $n!$ ("n factorial"):

Examples:

- Consider the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$. To use the Ratio Test, look at the limit:

  $$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{1}{2} = \frac{1}{2} < 1$$

  Hence, the series converges.

- Consider the series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$. To use the Ratio Test, look at the limit:

  $$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \frac{2^n}{n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{1}{2} = \lim_{n\to\infty} \frac{n+1}{2} \frac{1}{2} = \infty$$

  Hence, the series diverges.

- see Final Exam Review, #8(c)(d)

Checklist for infinite series:

To decide whether a given infinite series $\sum a_n$ converges or not, check the following:

- Is the series geometric? If so, you can determine convergence based on the value of r.
- Do the individual terms $a_n \to 0$ as $n \to \infty$? If not, the series diverges, by the Divergence (nth-Term) Test.
- Is the series a p-series? If so, you can determine convergence based on the value of p.
- Is a_n a ratio of two polynomials? Then look at the ratio of the leading terms and use the limit-comparison test with the appropriate p-series (or in the case the two polynomials have the same degree, the series will diverge by the Divergence Test).
- If none of the above apply, and especially if a_n involves n as an exponent and/or $n!$, try the Ratio Test.

For an alternating series $\sum(-1)^n a_n$:

- First analyze the non-alternating series $\sum a_n$:
- If $\sum a_n$ converges, the alternating series $\sum(-1)^n a_n$ is absolutely convergent.
- If $\sum a_n$ diverges, use the Alternating Series Test to check whether $\sum(-1)^n a_n$ is conditionally convergent.
- If the Alternating Series Test fails, it will usually be because $\lim_{n\to\infty} a_n \neq 0$, in which case the series is divergent by the Divergence Test.