MAT1575, Fall 2023 **Exam #2**
Instructor: Suman Ganguli **Nov 15, 2023** **Name:** ____________________

Question:	1	2	3	4	5	Total
Points:	10	10	10	5	15	50
Score:						

Show all your work and simplify your answers.

1. (10 points) Solve the following integrals using:

   (a) substitution:

$$\int \frac{\ln x}{x}\, dx =$$

Solution:

$$u = \ln x \Longrightarrow du = \frac{1}{x} dx$$

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

   (b) integration by parts:

$$\int xe^{2x}\, dx =$$

Solution: Integration by parts works with the following choices:

$$u = x \Longrightarrow du = dx$$

$$dv = e^{2x}\, dx \Longrightarrow v = \frac{e^{2x}}{2}$$

Then:

$$\int xe^{2x}\, dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\, dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

2. (10 points) Solve the following indefinite integrals using the method of partial fractions:

(a)

$$\int \frac{x-9}{x^2+3x-10}\,dx$$

Solution:

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2} \Longrightarrow x-9 = A(x-2)+B(x+5)$$

$$x=-5 \Longrightarrow -14 = -7A \Longrightarrow A = 2$$

$$x=2 \Longrightarrow -7 = 7B \Longrightarrow B = -1$$

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \left(\frac{2}{x+5} - \frac{1}{x-2}\right)dx = 2\ln|x+5| - \ln|x-2| + C$$

(b)

$$\int \frac{2x}{(x+2)^2}\,dx$$

Note: for this “repeated linear factor”, the partial fraction decomposition takes the form:

$$\frac{2x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Solution:

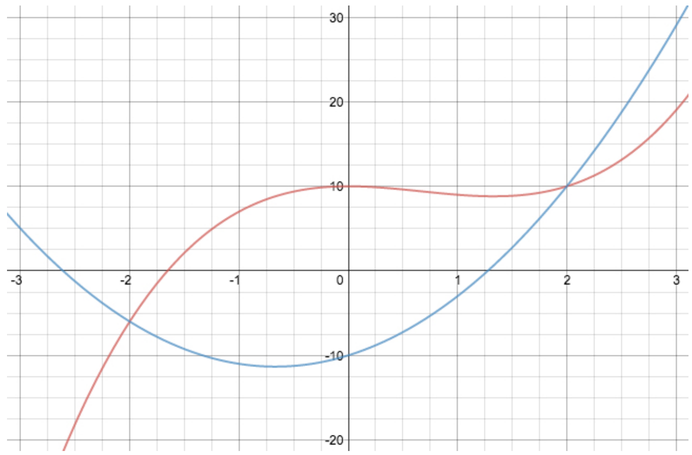
$$\frac{2x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$2x = A(x+2) + B$$

Substituting $x=-2$ yields $-4=B$. And by equating x-coefficients, we see that $A=2$.
Hence:

$$\int \frac{2x}{(x+2)^2}\,dx = \int \left(\frac{2}{x+2} - \frac{4}{(x+2)^2}\right)dx = 2\ln|x+2| + \frac{4}{(x+2)} + C$$

3. (10 points) Shown below are the graphs of $f(x) = x^3 - 2x^2 + 10$ and $g(x) = 3x^2 + 4x - 10$. Find the area of the region enclosed by the two graphs shown below:



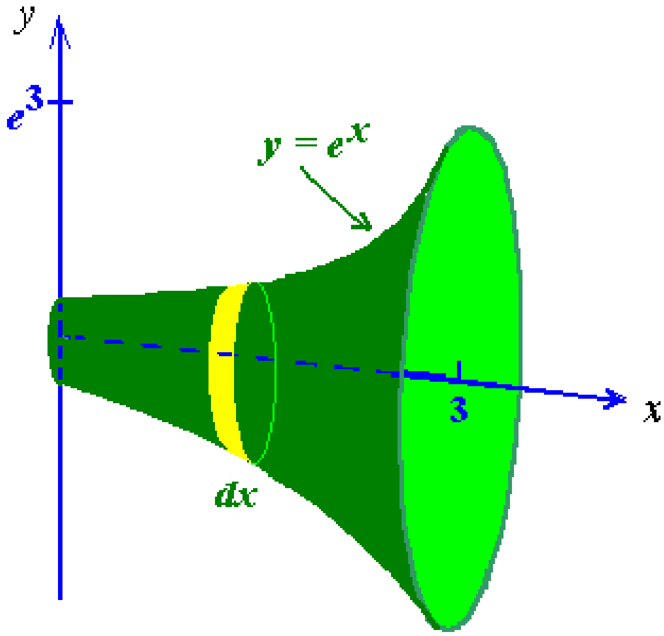
Use the fact that the two graphs intersect at $x = -2$ and $x = 2$ (i..e, you don't need to solve for these points algebraically.)

Solution:

$$\int_{-2}^{2} (x^3 - 2x^2 + 10) - (3x^2 + 4x - 10)\, dx = \int_{-2}^{2} (x^3 - 5x^2 - 4x + 20)\, dx =$$

$$\left[\frac{x^4}{4} - \frac{5x^3}{3} - 2x^2 + 20x\right]_{-2}^{2} = \left(4 - \frac{40}{3} - 8 + 40\right) - \left(4 + \frac{40}{3} - 8 - 40\right) = 80 - \frac{80}{3} = \frac{160}{3}$$

4. (5 points) Find the volume of the solid obtained by rotating the region under the graph of $y = e^x$ and above the interval $0 \leq x \leq 3$ around the x-axis:



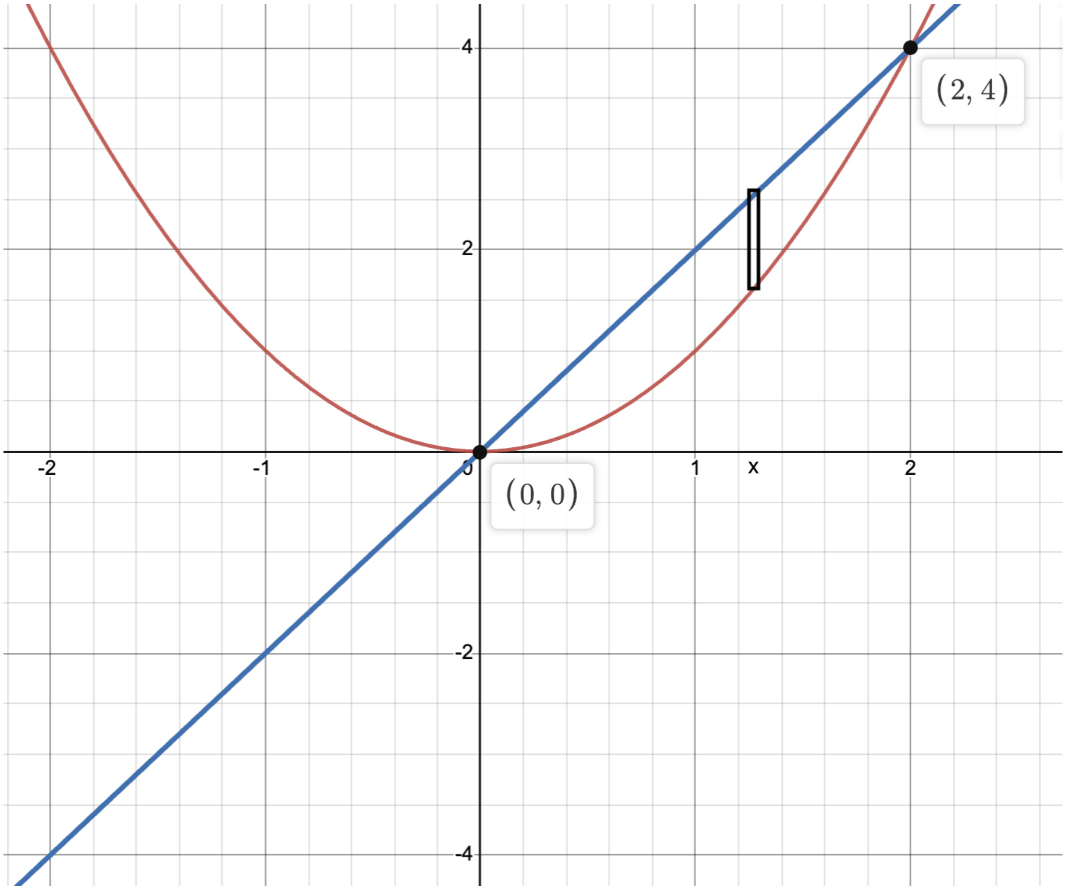
Hints:

- Set up an integral for the volume using the "disk method." Shown in the figure is a reprentative circular disk formed by taking a cross-sectional slice of width dx.
- The volume of such a disk is the circular cross-sectional area times dx.
- What is the circular cross-sectional area (as a function of x, for $0 \leq x \leq 3$)? That will be the integrand.
- Leave your answer in terms of e and π.

Solution:

$$V = \int_0^3 \pi(e^x)^2\, dx = \pi \int_0^3 e^{2x}\, dx = \frac{\pi}{2}\left[e^{2x}\right]_0^3 = \frac{\pi}{2}\left(e^6 - e^0\right) = \frac{\pi}{2}\left(e^6 - 1\right)$$

5. (15 points) Shown below are the graphs of $f(x) = x^2$ and $g(x) = 2x$.



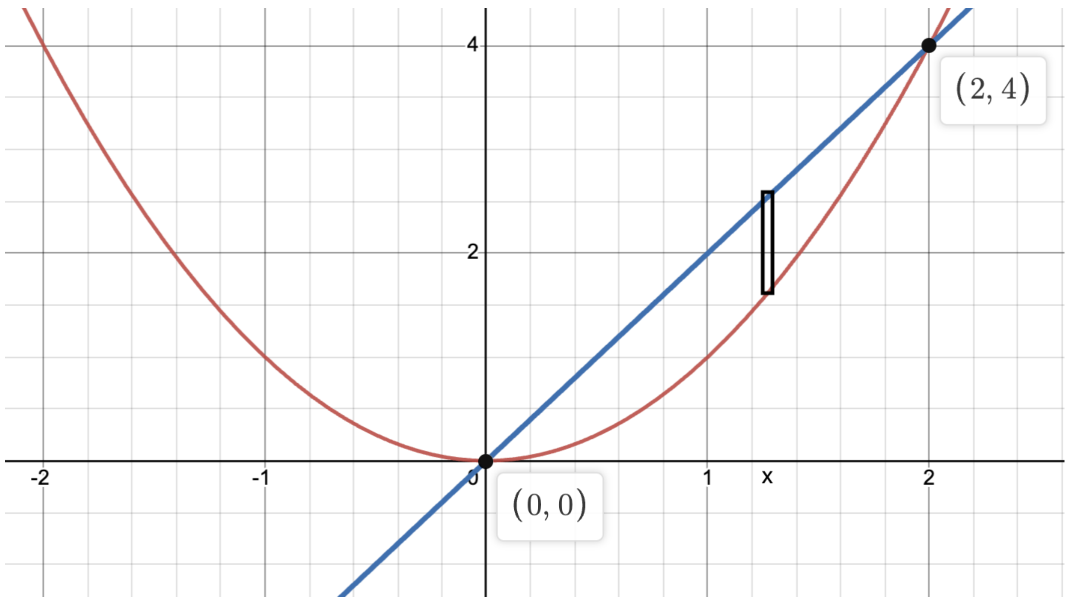
(a) Compute the volume of the solid obtained by rotating this region around the x-axis, using the washer method. First sketch a representative washer on the graph above, as generated by the rectangle of width dx shown on the graph. Label the "inner radius" and "outer radius" of the washer on the graph.
The sketch should help you set up the integral for the volume:

Volume =

Solution: Since $y = 2x$ forms the "outer radius" and $y = x^2$ forms the "inner radius" of each washer, the volume is:

$$\int_0^2 \pi(2x)^2 - \pi(x^2)^2 - \ dx = \pi \int_0^2 (4x^2 - x^4)\ dx = \pi \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right)$$

$$= 32\pi\left(\frac{1}{3} - \frac{1}{5}\right) = 32\pi\left(\frac{5}{15} - \frac{3}{15}\right) = \frac{64\pi}{15}$$



(b) Now find the volume of the solid obtained by rotating this region around the y-axis, using the method of cylindrical shells. Sketch a representative cylindrical shell on the graph above, as generated by the rectangle of width dx shown on the graph).

$$\text{Volume} = \int_0^2 2\pi x(2x - x^2)\, dx =$$

Solution:

$$\int_0^2 2\pi x(2x - x^2)\, dx = 2\pi \int_0^2 (2x^2 - x^3)\, dx = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= 2\pi \left(\frac{16}{3} - \frac{16}{4}\right) = 32\pi \left(\frac{1}{3} - \frac{1}{4}\right) = 32\pi \left(\frac{1}{12}\right) = \frac{8\pi}{3}$$

(c) Give a brief explanation of why the integrand in part (b) is $2\pi x(2x - x^2)\, dx$, i.e., why that expression gives the volume of a cylindrical shell for this solid of revolution. You may find it useful to refer to your sketch of a cylindrical shell above, labeling it with its dimensions.