MAT2440/D648 **Exam #2**
Instructor: Suman Ganguli **Due: Wednesday, April 13** **Name:** ______________________

Question:	1	2	3	4	5	Total
Points:	10	15	10	5	10	50
Score:						

1. (10 points) Recall the following definitions:

Definition: A real number x is *rational* if there exist integers a and b such that $x = \dfrac{a}{b}$. A real number is *irrational* if it not rational.

Definition: An integer n is *even* if $n = 2k$ for some integer k. An integer n is *odd* if $n = 2k + 1$ for some integer k.

Write out proofs of the following theorems. For each, clearly the state the assumption, the definition(s) used, and the conclusion, and show any necessary algebra.

a. **Theorem:** For any real number x, if x is rational and $x \neq 0$, then $1/x$ is also rational.
(Hint: Provide a direct proof, i.e., start by assuming a given real number x is rational and non-zero.)
Proof:

Solution: For a direct proof, we begin by assume the hypothesis is true. So a proof is as follows:
Assume that x is a non-zero rational real number. By definition (of being rational), $x = \dfrac{a}{b}$ for integers a and b. Then

$$\frac{1}{x} = \frac{1}{a/b} = \frac{b}{a}$$

So $1/x$ is also a ratio of two integers, and thus is also a rational number.

b. **Theorem:** For any integer n, if n^2 is even, then n must also be even.
(Hint: Provide a proof by contraposition. Hence, start by assuming a given integer n is *not* even, i.e., assume that n is odd.)

Solution: For a proof by contraposition, we begin by assuming the negation of the conclusion. So a proof is as follows:
Assume that a given integer n is *not* even, i.e., assume that n is odd. Then $n = 2k + 1$ for some integer k, and hence

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Thus, n^2 is also odd, since $n^2 = 2j + 1$ for an integer j (where $j = 2k^2 + 2k$).
This establishes the contrapositive of the theorem, and thus is a proof of the theorem.

2. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{0, 1\}$.

a. (5 points) List the elements of the following sets:

i. $A \cup B =$

Solution: $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$

ii. $A \cap B =$

Solution: $A \cap B = \{1\}$

iii. $A - B =$

Solution: $A - B = \{2, 3, 4, 5, 6\}$

iv. $B - A =$

Solution: $B - A = \{0\}$

v. $A \times B =$

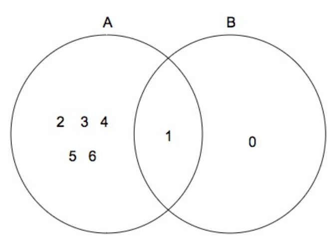
Solution: $A \times B = \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1), (5, 0), (5, 1), (6, 0), (6, 1)\}$

vi. (Extra credit!) $\mathcal{P}(A) \cap \mathcal{P}(B) =$

Solution: $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing, \{1\}\}$

b. (5 points) Draw a Venn diagram illustrating the sets A and B, representing all of their elements with points in the appropriate regions in the diagram.

Solution:



c. (5 points) Consider the function $f : A \times A \to \mathbb{N}$ defined by the formula $f(a_1, a_2) = a_1 + a_2$. (Note that we are still using $A = \{1, 2, 3, 4, 5, 6\}$.)

i. What is the range of f?

Solution: Note that the maximum value of f is $f(6, 6) = 6 + 6 = 12$, and the minimum value is $f(1, 1) = 1 + 1 = 2$. All the integers between 2 and 12 are also in the range, since each such integer is the sum of a pair of numbers from A. Thus, the range of f is the set { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }.

ii. Show that f is *not* a one-to-one function (i.e., find two distinct inputs in the domain $A \times A$ which get mapped by f to the same output in the range).

Solution: In order to show f is not one-to-one, we need to find two distinct inputs in the domain $A \times A$ which get mapped by f to the same output in the range. There are many such examples for this function; for instance $f(3, 1) = 3 + 1 = 4$ and $f(2, 2) = 2 + 2 = 4$.

3. Consider the following definition:

Definition: If a and b are integers, we say that a *divides* b if there is an integer j such that $b = a * j$, or equivalently, if $\frac{b}{a}$ is an integer j. We also say a is a *factor* of b, and b is a *multiple* of a.

Examples: 3 divides 12 since $12/3 = 4$ is an integer, i.e., $12 = 3 * j$ for $j = 4$. But 5 does not divide 12, since $12/5$ is not an integer.

a. (5 points) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. List the elements of the following subsets of U:

i. $A = \{n \in U \mid 3 \text{ divides } n\} = \{n \in U \mid n \text{ is a multiple of } 3\} =$

Solution: A consists of the integer multiples of 3 in U, i.e., $A = \{3, 6, 9, 12\}$

ii. $B = \{n \in U \mid n \text{ divides } 12\} = \{n \in U \mid n \text{ is a factor of } 12\} =$

Solution: B consists of all the factors of 12 in U, i.e., $B = \{1, 2, 3, 4, 6, 12\}$

b. (5 points) Provide a proof of the following theorem:

Theorem: If a divides b and a divides c, then a also divides $b + c$.

(Hint: Provide a direct proof, i.e., start by assuming a divides b and a divides c. Then *apply the definition* given above.)

Proof:

Solution: We use the notation $a \mid b$ for "a divides b" (this is standard notation that we will introduce in later in the semester, when we study number theory; see Section 4.1 of the textbook.)
For a direct proof of the theorem, assume $a \mid b$ and $a \mid c$. Then, by the definition, $b = a * j$ and $c = a * k$ for some integers j and k. Then $b + c = a * j + a * k = a * (j + k)$, which establishes that $a \mid (b + c)$.

4. (5 points) Consider the following definition of a *one-to-one* function:

Definition: A function $f : A \longrightarrow B$ is *one-to-one* if and only if for any $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$.
We can translate this definition into predicate logic as follows:

$$\forall a_1 \forall a_2 \left[(a_1 \in A \land a_2 \in A \land a_1 \neq a_2) \longrightarrow f(a_1) \neq f(a_2)\right]$$

a. What is the definition of an *onto* function? Give the definition in natural language, i.e., using words (as in the textbook!)

Definition: A function $f : A \longrightarrow B$ is *onto* if and only if ...

Solution: ... for every element b of the co-domain B, there is an element a of the domain A such that $f(a) = b$.

b. Now translate your natural language definition of f being onto into a statement of predicate logic, i.e., using quantifiers and logical connectives:

Solution:

$$\forall b \in B\, \exists a \in A\, [f(a) = b]$$

5. (10 points) Let $A = \{a, b, c\}$.

a. List the elements of the power set of A. (Hint: Since A has 3 elements, its power set has $2^3 = 8$ elements.)

$\mathcal{P}(A) =$

Solution: $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

b. Construct a function $f : A \to \mathcal{P}(A)$ which has the following properties:

1. f is one-to-one, and
2. $\forall x \in A\, (x \in f(x))$

Solution: In order to make f one-to-one need to choose 3 *different* elements from $\mathcal{P}(A)$ (i.e., subsets of A) as the outputs of $f(a), f(b)$ and $f(c)$.

In order to satisfy the 2nd condition, for $f(a)$ we need to choose a subset of A containing a. There are three such subsets of A: $\{a\}, \{a, b\}, \{a, b, c, \}$. In order to define f, we need choose one of them. Let's say we choose $\{a\}$, i.e., we set $f(a) = \{a\}$. Then for $f(b)$ choose a subset of A containing b, and for $f(c)$ choose a subset of A containing c. A natural set of choices is

$f(a) = \{a\}$

$f(b) = \{b\}$

$f(c) = \{c\}$

But there are other correct solutions, e.g.,

$f(a) = \{a, b\}$ $f(b) = \{b, c\}$ $f(c) = \{a, b, c\}$

An interesting question is: how many different correct solutions are there, i.e., how many different such functions $f : A \to \mathcal{P}(A)$ satisfy those two properties?