MAT2440/D648 **Exam #2**
Instructor: Suman Ganguli **Due: Wednesday, April 13** **Name:** ____________________

Question:	1	2	3	4	5	Total
Points:	10	15	10	5	10	50
Score:						

1. (10 points) Recall the following definitions:

   Definition: A real number x is *rational* if there exist integers a and b such that $x = \frac{a}{b}$. A real number is *irrational* if it not rational.

   Definition: An integer n is *even* if $n = 2k$ for some integer k. An integer n is *odd* if $n = 2k + 1$ for some integer k.

   Write out proofs of the following theorems. For each, clearly the state the assumption, the definition(s) used, and the conclusion, and show any necessary algebra.

   a. **Theorem:** For any real number x, if x is rational and $x \neq 0$, then $1/x$ is also rational.
   (Hint: Provide a direct proof, i.e., start by assuming a given real number x is rational and non-zero.)
   Proof:

   b. **Theorem:** For any integer n, if n^2 is even, then n must also be even.
   (Hint: Provide a proof by contraposition. Hence, start by assuming a given integer n is *not* even, i.e., assume that n is odd.)

2. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{0, 1\}$.
   a. (5 points) List the elements of the following sets:
      i. $A \cup B =$
      ii. $A \cap B =$
      iii. $A - B =$
      iv. $B - A =$
      v. $A \times B =$
      vi. (Extra credit!) $\mathcal{P}(A) \cap \mathcal{P}(B) =$
   b. (5 points) Draw a Venn diagram illustrating the sets A and B, representing all of their elements with points in the appropriate regions in the diagram.
   c. (5 points) Consider the function $f : A \times A \to \mathbb{N}$ defined by the formula $f(a_1, a_2) = a_1 + a_2$. (Note that we are still using $A = \{1, 2, 3, 4, 5, 6\}$.)
      i. What is the range of f?
      ii. Show that f is *not* a one-to-one function (i.e., find two distinct inputs in the domain $A \times A$ which get mapped by f to the same output in the range).

3. Consider the following definition:

Definition: If a and b are integers, we say that a *divides* b if there is an integer j such that $b = a * j$, or equivalently, if $\frac{b}{a}$ is an integer j. We also say a is a *factor* of b, and b is a *multiple* of a.

Examples: 3 divides 12 since $12/3 = 4$ is an integer, i.e., $12 = 3 * j$ for $j = 4$. But 5 does not divide 12, since $12/5$ is not an integer.

a. (5 points) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. List the elements of the following subsets of U:

i. $A = \{n \in U \mid 3 \text{ divides } n\} = \{n \in U \mid n \text{ is a multiple of } 3\} =$

ii. $B = \{n \in U \mid n \text{ divides } 12\} = \{n \in U \mid n \text{ is a factor of } 12\} =$

b. (5 points) Provide a proof of the following theorem:

Theorem: If a divides b and a divides c, then a also divides $b + c$.

(Hint: Provide a direct proof, i.e., start by assuming a divides b and a divides c. Then *apply the definition* given above.)

Proof:

4. (5 points) Consider the following definition of a *one-to-one* function:

Definition: A function $f : A \longrightarrow B$ is *one-to-one* if and only if for any $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$.

We can translate this definition into predicate logic as follows:

$$\forall a_1 \forall a_2 \left[(a_1 \in A \wedge a_2 \in A \wedge a_1 \neq a_2) \longrightarrow f(a_1) \neq f(a_2)\right]$$

a. What is the definition of an *onto* function? Give the definition in natural language, i.e., using words (as in the textbook!)

Definition: A function $f : A \longrightarrow B$ is *onto* if and only if . . .

b. Now translate your natural language definition of f being onto into a statement of predicate logic, i.e., using quantifiers and logical connectives:

5. (10 points) Let $A = \{a, b, c\}$.

a. List the elements of the power set of A. (Hint: Since A has 3 elements, its power set has $2^3 = 8$ elements.)

$\mathcal{P}(A) =$

b. Construct a function $f : A \to \mathcal{P}(A)$ which has the following properties:

1. f is one-to-one, and
2. $\forall x \in A\, (x \in f(x))$

$f(a) =$

$f(b) =$

$f(c) =$