Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements:

$$\neg\forall\epsilon>0\ \exists\delta>0\ \forall x(0<|x-a|<\delta \rightarrow |f(x)-L|<\epsilon)$$
$$\equiv \exists\epsilon>0\ \neg\exists\delta>0\ \forall x(0<|x-a|<\delta \rightarrow |f(x)-L|<\epsilon)$$
$$\equiv \exists\epsilon>0\ \forall\delta>0\ \neg\forall x(0<|x-a|<\delta \rightarrow |f(x)-L|<\epsilon)$$
$$\equiv \exists\epsilon>0\ \forall\delta>0\ \exists x\ \neg(0<|x-a|<\delta \rightarrow |f(x)-L|<\epsilon)$$
$$\equiv \exists\epsilon>0\ \forall\delta>0\ \exists x(0<|x-a|<\delta \wedge |f(x)-L|\geq\epsilon).$$

In the last step we used the equivalence $\neg(p \rightarrow q) \equiv p \wedge \neg q$, which follows from the fifth equivalence in Table 7 of Section 1.3.

Because the statement "$\lim_{x\rightarrow a} f(x)$ does not exist" means for all real numbers L, $\lim_{x\rightarrow a} f(x) \neq L$, this can be expressed as

$$\forall L \exists\epsilon>0\ \forall\delta>0\ \exists x(0 < |x-a| < \delta \wedge |f(x)-L| \geq \epsilon).$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x-a| < \delta$ and $|f(x)-L| \geq \epsilon$. ◀

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\forall x\exists y(x < y)$

b) $\forall x\forall y(((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$

c) $\forall x\forall y\exists z(xy = z)$

2. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\exists x\forall y(xy = y)$

b) $\forall x\forall y(((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$

c) $\forall x\forall y\exists z(x = y + z)$

3. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x\exists yQ(x, y)$ **b)** $\exists x\forall yQ(x, y)$

c) $\forall x\exists yQ(x, y)$ **d)** $\exists y\forall xQ(x, y)$

e) $\forall y\exists xQ(x, y)$ **f)** $\forall x\forall yQ(x, y)$

4. Let $P(x, y)$ be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x\exists yP(x, y)$ **b)** $\exists x\forall yP(x, y)$

c) $\forall x\exists yP(x, y)$ **d)** $\exists y\forall xP(x, y)$

e) $\forall y\exists xP(x, y)$ **f)** $\forall x\forall yP(x, y)$

5. Let $W(x, y)$ mean that student x has visited website y, where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

a) W(Sarah Smith, www.att.com)

b) $\exists xW(x,$ www.imdb.org)

c) $\exists yW$(José Orez, y)

d) $\exists y(W$(Ashok Puri, $y) \wedge W$(Cindy Yoon, y))

e) $\exists y\forall z(y \neq$ (David Belcher) $\wedge$ (W(David Belcher, $z) \rightarrow W(y,z)$))

f) $\exists x\exists y\forall z((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

6. Let $C(x, y)$ mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

a) C(Randy Goldberg, CS 252)

b) $\exists xC(x,$ Math 695)

c) $\exists yC$(Carol Sitea, y)

d) $\exists x(C(x,$ Math 222) $\wedge$ $C(x,$ CS 252))

e) $\exists x\exists y\forall z((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$

f) $\exists x\exists y\forall z((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

7. Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T$(Abdallah Hussein, Japanese)

b) $\exists x T(x, \text{Korean}) \wedge \forall x T(x, \text{Mexican})$
c) $\exists y(T(\text{Monique Arsenault}, y) \vee T(\text{Jay Johnson}, y))$
d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$
e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$
f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

8. Let $Q(x, y)$ be the statement "Student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.
b) No student at your school has ever been a contestant on a television quiz show.
c) There is a student at your school who has been a contestant on *Jeopardy!* and on *Wheel of Fortune*.
d) Every television quiz show has had a student from your school as a contestant.
e) At least two students from your school have been contestants on *Jeopardy!*.

9. Let $L(x, y)$ be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lydia does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom Lynn loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

10. Let $F(x, y)$ be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Fred.
b) Evelyn can fool everybody.
c) Everybody can fool somebody.
d) There is no one who can fool everybody.
e) Everyone can be fooled by somebody.
f) No one can fool both Fred and Jerry.
g) Nancy can fool exactly two people.
h) There is exactly one person whom everybody can fool.
i) No one can fool himself or herself.
j) There is someone who can fool exactly one person besides himself or herself.

11. Let $S(x)$ be the predicate "x is a student," $F(x)$ the predicate "x is a faculty member," and $A(x, y)$ the predicate "x has asked y a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois has asked Professor Michaels a question.
b) Every student has asked Professor Gross a question.
c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

12. Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

a) Jerry does not have an Internet connection.
b) Rachel has not chatted over the Internet with Chelsea.
c) Jan and Sharon have never chatted over the Internet.
d) No one in the class has chatted with Bob.
e) Sanjay has chatted with everyone except Joseph.
f) Someone in your class does not have an Internet connection.
g) Not everyone in your class has an Internet connection.
h) Exactly one student in your class has an Internet connection.
i) Everyone except one student in your class has an Internet connection.
j) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
l) There are two students in your class who have not chatted with each other over the Internet.
m) There is a student in your class who has chatted with everyone in your class over the Internet.
n) There are at least two students in your class who have not chatted with the same person in your class.
o) There are two students in the class who between them have chatted with everyone else in the class.

13. Let $M(x, y)$ be "x has sent y an e-mail message" and $T(x, y)$ be "x has telephoned y," where the domain consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)

a) Chou has never sent an e-mail message to Koko.
b) Arlene has never sent an e-mail message to or telephoned Sarah.
c) José has never received an e-mail message from Deborah.
d) Every student in your class has sent an e-mail message to Ken.
e) No one in your class has telephoned Nina.