#5. (a) If A & B are both $m\times n$, then both AB^T and A^TB are defined.

True

$A \equiv m\times n \quad B \equiv m\times n$

$A^T \equiv n\times m \quad B^T \equiv n\times m.$

$A_{m\times n} \quad B^T_{n\times m}$ # Columns of A = # rows of B^T

$\Downarrow$

AB^T possible

$A^T_{n\times m} \quad B_{m\times n}$ # Columns of A^T = # rows of B

$\Downarrow$

A^TB possible.

(b) $AB = C$ given

False

C has 2 columns.

Note: If A is an $m\times n$ matrix & B is an $n\times p$ matrix then AB should be a $m\times p$ matrix.

So if C has 2 columns then B must have 2 columns.

So A doesn't necessarily have to have 2 columns.

You can also provide a counterexample to show that this statement is false.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad C = AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2+1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

C has

$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}_{1\times 3} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$

$\underset{1\times 2}{C = AB} = \begin{pmatrix} -1+0+1 & 0+1+3 \end{pmatrix} = \begin{pmatrix} 0 & 4 \end{pmatrix}$

Here $AB = C$

C has 2 columns but A has 3 columns.

$\Rightarrow$ the statement is false.

(c)

False $BC = BD$ (Given)

Let $B = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Then $BC = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad BD = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Here $BC = BD$

but $C \neq D$

So the statement is false.

5(d) If A & B are $n\times n$ then

$(A+B)(A-B) = A^2 - B^2$

False

$(A+B)(A-B)$

$= (A+B)A - (A+B)B = A^2 + BA - AB - B^2$

Therefore $(A+B)(A-B) = A^2 - B^2$ only if $AB = BA$ i.e

A & B are Commutative.

#6 $A = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{pmatrix}$

(a) $\underline{A^{-1}}$

$A \mid I$

$$= \left(\begin{array}{ccc|ccc} 1 & 3 & 8 & 1 & 0 & 0 \\ 2 & 4 & 11 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 8 & 1 & 0 & 0 \\ 0 & -2 & -5 & -2 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{2R_3 - R_2} \begin{pmatrix} 1 & 3 & 8 & 1 & 0 & 0 \\ 0 & -2 & -5 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 1 & 3 & -1 & 1 & 0 \\ 0 & -2 & -5 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{2R_1 + R_2} \begin{pmatrix} 2 & 0 & 1 & -4 & 3 & 0 \\ 0 & -2 & -5 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow[R_1 + R_3]{R_2 - 5R_3} \begin{pmatrix} 2 & 0 & 0 & -4 & 2 & 2 \\ 0 & -2 & 0 & -2 & 6 & -10 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{-2}R_2,\ -1R_3]{\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right)$$

$$\qquad\qquad I \qquad\qquad\qquad A^{-1}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 5 \\ \cancel{4}0 & 1 & -2 \end{pmatrix}$$

<u>Check.</u> $$AA^{-1} = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 5 \\ \cancel{4}0 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -2+3+0 & 1-9+8 & 1+15-16 \\ -\cancel{4}+\cancel{4}+0 & 2-12+11 & 2+20-22 \\ -\cancel{2}+\cancel{2}+0 & 1-6+5 & 1+\cancel{10}-\cancel{10} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cancel{9}-\cancel{9} & \cancel{16}-\cancel{16} \\ 0 & 13-12 & \cancel{22}-\cancel{22} \\ 0 & \cancel{6}-\cancel{6} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\Rightarrow A^{-1}$ is <u>correct.</u>

(b) AB can be computed since # of cols. of A = # of rows of B.
BA can't be computed since # of cols. of B = 2 $\neq$ # of rows of A = 3

6 (c) Compute AB & A^TB.

$$AB = \begin{pmatrix} \cancel{-3}+\cancel{3}+24 & 5+15+32 \\ -6+4+33 & 10+20+44 \\ -3+2+15 & 5+10+20 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 52 \\ 37 & 74 \\ 17 & 35 \end{pmatrix}$$

$$\underset{3\times2}{A^TB} = \underset{3\times3}{\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 8 & 11 & 5 \end{pmatrix}} \underset{3\times2}{\begin{pmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{pmatrix}}$$

$$= \begin{pmatrix} \cancel{-3}+2+\cancel{3} & 5+10+4 \\ -9+4+6 & 15+20+8 \\ -24+11+15 & 40+55+20 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 19 \\ 1 & 43 \\ 2 & 115 \end{pmatrix}$$

#9 $A^2 - 2A + I = 0$

$\Rightarrow A(A^2 - 2A + I) = A.O$

$\Rightarrow A^3 - 2A^2 + \underbrace{AI}_{A} = 0$

$\Rightarrow A^3 = 2A^2 - A$

$\Rightarrow$ [illegible] A since $IA = A$.

$\Rightarrow$ [illegible]

$=$ [illegible]

$\Rightarrow A^3 = 2(2A - I) - A$ since $A^2 = 2A - I$

$= 4A - 2I - A$

$\boxed{A^3 = 3A - 2I}$ proved.

$A^4 = \cancel{A(3}\, A.A^3$

$= A(3A - 2I)$

$= 3A^2 - 2AI$

$= 3A^2 - 2A$

$= 3(2A - I) - 2A$ since $A^2 = 2A - I$

$= 6A - 3I - 2A$

$\boxed{A^4 = 4A - 3I}$ proved.

#10

$x \longrightarrow Ax$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $\begin{pmatrix} 2 \\ 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Since $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is $\cancel{2}\times 1$ and the output $Ax = \underset{2\times 1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is

$\Rightarrow$ A must be of dimension 2×2.

Let A be $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} a_{11} + 3a_{12} = 1 \quad \text{— (1)} \\ a_{21} + 3a_{22} = 1 \quad \text{— (2)} \end{array}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\Rightarrow$ $2a_{11} + 7a_{12} = 3$ — (3)

$2a_{21} + 7a_{22} = 1$ — (4)

Solving (1), (2), (3), (4) should give us A. $A \mid \vec{b}$

$$\left.\begin{array}{l} a_{11} + 3a_{12} = 1 \\ a_{21} + 3a_{22} = 1 \\ 2a_{11} + 7a_{12} = 3 \\ 2a_{21} + 7a_{22} = 1 \end{array}\right\} \Rightarrow \begin{array}{c} \begin{array}{cccc} a_{11} & a_{12} & a_{21} & a_{22} \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 2 & 7 & 0 & 0 & 3 \\ 0 & 0 & 2 & 7 & 1 \end{array}\right) \end{array}$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{array}{l} a_{11} = -2 \\ a_{12} = 1 \\ a_{21} = 4 \\ a_{22} = -1 \end{array}$$

$$\boxed{A = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}}$$

Checking:

$$\begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2+3 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -4+7 \\ 8-7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

CORRECT!