Final Exam Review

#(1) (i) $\det(AB) = \det(A)\det(B) = (-2)(3) = \boxed{-6}$ Ans.

(ii) $\det(5A^{-1}) = (5)^4 \det(A^{-1})$ (since $A \equiv 4\times4$)

$= 5^4 \cdot \frac{1}{\det A} = \boxed{\frac{5^4}{-2}}$ Ans.

(iii) $\det(B^T) = \det(B) = \boxed{3}$ Ans.

(iv) $\det(B^T A^{-1}) = \det(B^T)\det(A^{-1}) = \det(B) \cdot \frac{1}{\det(A)}$

$= \boxed{\frac{3}{-2}}$ Ans.

(v) $\det(B^{10}) = \underbrace{\det(B) \cdot \det(B) \cdots \det(B)}_{10 \text{ terms}} = (\det B)^{10} = (3)^{10}$

$= \boxed{3^{10}}$

#(2) $$H = \left\{ \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

$$\begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_2 \\ x_2 \end{bmatrix}$$

Method I

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} x_2 \quad x_1, x_2 \in \mathbb{R}$$

x_1, x_2 are reals #'s.

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow H = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ are in $\mathbb{R}^3$, and $H = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

$\Rightarrow$ by theorem H is a subspace of $\mathbb{R}^3$

Method II (a) Is $\vec{0}$ a vector in H?

Take $x_1, x_2 = 0$.

Then $\begin{pmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in H ✓ ~~pro~~ 1st criterion of subspace is verified

(b) Let $\vec{u} = \begin{pmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} y_1 \\ y_1 - y_2 \\ y_2 \end{pmatrix}$, x_1, y_1, x_2, y_2 are ~~in R~~ real #s.

$\Rightarrow \vec{u} + \vec{v} = \begin{pmatrix} x_1 + y_1 \\ (x_1 + y_1) - (x_2 + y_2) \\ x_2 + y_2 \end{pmatrix}$

Since $x_1 + y_1$ is in real #'s
$x_2 + y_2$ " " " "

$\Rightarrow \vec{u} + \vec{v}$ is also in H.
2nd ~~pro~~ criterion is verified. ✓

(c) Let $\vec{u} = \begin{pmatrix} x_1 \\ x_1 - x_2 \\ x_2 \end{pmatrix}$, x_1, x_2 are real #s.

Let c be a real #s.

$c\vec{u} = \begin{pmatrix} cx_1 \\ cx_1 - cx_2 \\ cx_2 \end{pmatrix}$ cx_1, cx_2 are real #s.

$\Rightarrow c\vec{u}$ is also in H
$\Rightarrow$ Third criterion is verified ✓

$\Rightarrow H$ is a subspace of $\mathbb{R}^3$.

~~#~~ 3(b) Basis of $H = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

3(c) The plane must pass through the origin

$\Rightarrow \quad ax + by + cz = 0$

Taking $(x=1,\ y=1,\ z=0)$ ← the 1st basis vector values

$a + b + 0 = 0 \Rightarrow \boxed{b = -a}$

$\Rightarrow \quad ax - ay + cz = 0$

Taking $(x=0,\ y=-1,\ z=1)$ ← the 2nd basis vector values

$0 + a + c = 0 \Rightarrow \boxed{c = -a}$

The equ. becomes,

$ax + by + cz = 0$

$\Rightarrow ax - ay - az = 0$

$a(x - y - z) = 0, \quad a \neq 0$

$x - y - z = 0$ (Dividing both sides by a)

$\Rightarrow \boxed{z = x - y}$

$\Rightarrow$ The basis vector of H spans the ~~space~~ plane $z = x - y$, ~~gen~~ (a plane generated by the basis vector)

#3 $\left(\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left(\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{cccccc} 1 & 0 & 3 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 + R_3} \left(\begin{array}{cccccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{-3} R_3} \left(\begin{array}{cccccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & -1/3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -1/3 \end{pmatrix}$$

3(b)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\vec{b}$$

$$= \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -1/3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 - 2 - 2 \\ 1 + 1 + 0 \\ \cancel{-\frac{2}{3}} + 0 + \cancel{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}$$

$$\boxed{x_1 = -5, \quad x_2 = 2, \quad x_3 = 0}$$

#4

$$\begin{pmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 3 & 6 & -5 & 4 & 0 & 0 \\ 1 & 2 & 0 & 3 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot column   pivots.   pivot column

Therefore the basis for the column space of ~~M~~ is the ~~corresponding~~ columns of M corresponding to the pivot columns.

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ 0 \end{pmatrix} \right\}$$

(b) For Null space we need to solve.

$$M\vec{x} = \vec{0}$$

$$\xrightarrow{R_1 + 2R_2} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_3 + x_4 = 0 \Rightarrow \boxed{x_3 = -x_4}$$

$$x_1 + 2x_2 + 3x_4 = 0$$

$$x_1 = -2x_2 - 3x_4$$

x_5, x_6 can be anything.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -x_4 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_5 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_6$$

Basis for Nullspace

$$N(M) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right.$$

#5 $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

(a) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0$

$\boxed{\lambda = 5, \quad \lambda = -1}$ $\Rightarrow$ distinct eigenvalues

$\Rightarrow$ 2 indept. eigenvector

For $\lambda = 5$ $\begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$

$\xrightarrow{2R_2 + R_1} \begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{4}R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$x_1 = x_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, x_2 free variable, $x_2 = 1$

For $\lambda = -1$ $\begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$

$\xrightarrow{R_2 - R_1} \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$, x_2 free var.

$\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $x_2 = 1$, 5(b) $P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ $\quad P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

$$= \frac{1}{1+2} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$A = PDP^{-1}$$

$$A^5 = PD^5P^{-1} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^5 & 0 \\ 0 & (-1)^5 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3125 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}}$$

multiply these 3 matrices to get A^5.

6(a) If three vectors in $\mathbb{R}^3$ touch a line that does not pass through the origin, then all these vectors are in the same plane in $\mathbb{R}^3$. Since any 2 vectors are linearly independent since they are not multiple of each other (given) $\Rightarrow$ therefore ~~then~~ they span a plane in $\mathbb{R}^3$ generated by these 2 vectors.

Therefore, the 3rd vector which also lies on this plane must be a linear combination of the other 2 vectors $\Rightarrow$ the vectors are linearly dependent.

6(b) Note: None of these vectors are multiple of each other.

The Only way these 2 vectors can be linearly indept. if they are multiple of each other which is not the case here $\Rightarrow$ the vector will be linearly indept. (Note that none of these vectors are $\vec{0}$ vector).

the first row, first column element

#8

$$\det(A) = 1 \cdot \begin{vmatrix} 8 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & \frac{1}{3} & \pi & 1 \end{vmatrix}$$

$$(-1)^{1+2}(-5) \begin{vmatrix} -5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & \frac{1}{3} & \pi & 1 \end{vmatrix}$$

$$\Rightarrow \det(A) = (1)(8) \begin{vmatrix} 6 & 0 & 0 \\ 4 & -3 & 0 \\ \frac{1}{3} & \pi & 1 \end{vmatrix} + 5(-5) \begin{vmatrix} 6 & 0 & 0 \\ 4 & -3 & 0 \\ \frac{1}{3} & \pi & 1 \end{vmatrix}$$

$$= (8)(6) \begin{vmatrix} -3 & 0 \\ \pi & 1 \end{vmatrix} - 0 + 0$$

$$-(25)(6) \begin{vmatrix} -3 & 0 \\ \pi & 1 \end{vmatrix} - 0 + 0$$

$$= (48)(-3-0) - 150(-3-0)$$

$$= -(3)(48) + (3)(150)$$

$$= \boxed{306}$$

#9. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 2/4 & 0 \\ 0 & 2/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$(A+B) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$(A+B)^{-1} = \frac{1}{\det(A+B)} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/9 & 0 \\ 0 & 3/9 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

$$A^{-1} + B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

$$(A+B)^{-1} \neq A^{-1} + B^{-1}$$

9(b) We've done this type of problem many times in classroom.

#10. (a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

By theorem any set with zero vector is linearly dependent

since $c_1 \vec{0} + \cancel{c_2}\, 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = 0$.

10(b) $\vec{u}_1 = \pi \vec{u}_2$ linearly dept.

since $\vec{u}_2$ is a scalar multiple of $\vec{u}_1$.

11. (a) Given $B = PAP^{-1}$

$$\det(B) = \det(P)\det(A)\det(P^{-1})$$
$$= \det(P)\det(P^{-1})\det(A)$$
$$= \det(PP^{-1})\det(A)$$
$$= \det(I)\det(A)$$
$$= \det(A) \checkmark \text{ proved.}$$

11(b) $B = PAP^{-1}$

$B - \lambda I_n = PAP^{-1} - \lambda I_n$

$= PAP^{-1} - \cancel{\lambda P P^{-1} I_n}\ \lambda P I_n P^{-1}$

$= \cancel{P(A-}$

$= PAP^{-1} = P(\lambda I_n) P^{-1}$

$= P(A - \lambda I_n) P^{-1}$

$\det(B - \lambda I_n) = \det(P) \det(A - \lambda I_n) \det(P^{-1})$

$= \det(P) \det(P^{-1}) \det(A - \lambda I_n)$

$= \det(P \cdot P^{-1}) \det(A - \lambda I_n)$

$= \det(I) \det(A - \lambda I_n)$

$= \det(A - \lambda I_n)$ ✓ proved

11(c) $B = PAP^{-1}$

$$B^n = \underbrace{(PAP^{-1})(PAP^{-1}) \cdots (PAP^{-1})}_{n \text{ times}}$$

$= PA(P^{-1}P) A(P^{-1}P) \cdots PA(P^{-1}P)AP^{-1}$

$= PA(I)A(I) \cdots PA(I)AP^{-1}$

$= PAA \cdots AAP^{-1}$

$= PA^nP^{-1}$ ✓ proved

#12. Skip