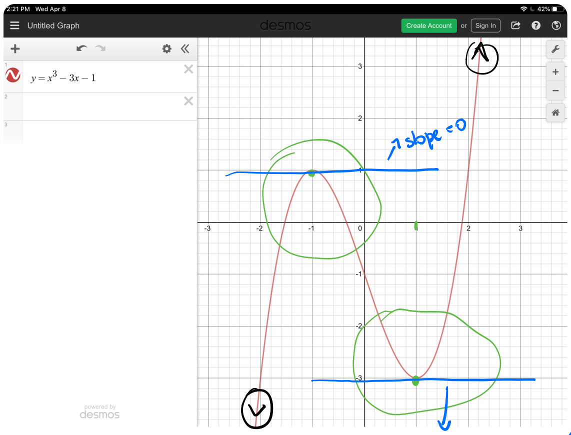



slope 0 tangent line at x=1

$x=1$ local minimum

$x=-1$ local maximum

$f(x) = x^3 - 3x - 1$

$f'(1) = 0$

$\underbrace{f'(-1)}$ = 0

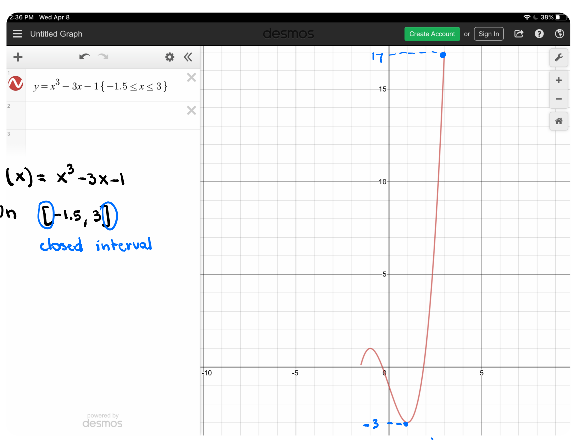
slope of the tangent line at $x=-1$

horizontal line has slope 0

$f(x) = x^3 - 3x - 1$ domain $(-\infty, \infty)$

There is no "highest" point (NO ABSOLUTE MAXIMUM)

There is no "lowest" point (NO ABSOLUTE MINIMUM)




$f(x) = x^3 - 3x - 1$

on $[-1.5, 3]$

closed interval

↗ critical point

f has an absolute min at $x=1$

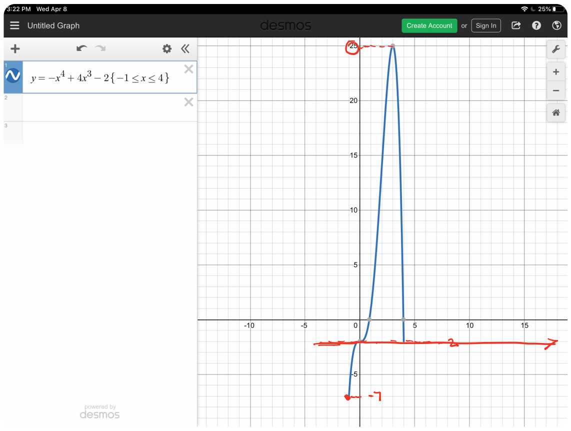
$-3$ is the ABSOLUTE MINIMUM of f

f has an absolute max at $x=3$ ↗ end of the interval

17 is the absolute max of f

Absolute max and absolute min of f are called EXTREME VALUES of f

or absolute extrema




x=0 critical
no local max
no local min
horizontal tangent line

# MAXIMA AND MINIMA

**DEF** Let $c$ be in the domain of a function $f$.
We say that $c$ is a CRITICAL POINT of $f$ if $f'(c)=0$ or if $f'(c)$ is undefined.

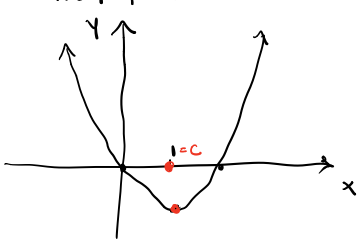
**EX** $f(x)=x^2-2x$

$f'(x)=2x-2=0$

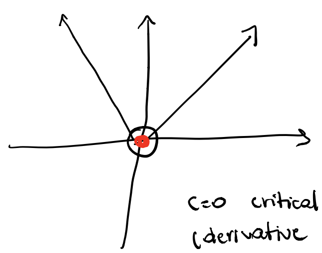
$2x-2=0$

$\frac{2x}{2}=\frac{2}{2}$ $\quad x=1$

critical point $c=1$




$f(x)=|x|$



$c=0$ critical point
(derivative DNE)

**EX** Find the critical points of $f(x)=x^3-3x-1$

x=1
x=-1

$f'(x)=0$

$f'(x)=3x^2-3=0$ $\qquad 3x^2=3$

$x^2=1 \qquad x=\pm\sqrt{1}=\pm 1$

$3(x^2-1)=0$

$3(x-1)(x+1)=0 \qquad \begin{matrix} x-1=0 \\ x+1=0 \end{matrix} \qquad \boxed{\begin{matrix} x=1 \\ x=-1 \end{matrix}}$

GRAPH ABOVE

**THEOREM (FERMAT'S)** If $f$ has a local max or a local min at $c$ and $f$ is differentiable at $c$, then $f'(c)=0$ and so $c$ is a critical point of $f$.

## EXTREME VALUES THEOREM

If $f$ is continuous over a closed interval $[a,b]$, then $f$ has an ABSOLUTE MAX $f(c)$ and an ABSOLUTE MIN $f(d)$ at some numbers $c, d$ in $[a,b]$.

To find absolute max and absolute min:

1. Find $f(a)$ and $f(b)$
2. Find all critical points of $f$ that are in $(a,b)$ and evaluate $f$ at those critical points.
3. The largest of the values from steps 1 and 2 is the absolute max of $f$, the smallest is the absolute min of $f$

**Ex** Find the absolute max and absolute min of

$$f(x) = -x^4 + 4x^3 - 2 \quad \text{on } [\underset{\text{"}a\text{"}}{-1}, \underset{\text{"}b\text{"}}{4}]$$

| x | f(x) |
|---|---|
| -1 | -7 |
| 4 | -2 |
| 0 | -2 |
| 3 | 25 |

ABSOLUTE MIN

ABSOLUTE MAX

1. $a = -1 \qquad f(-1) = -(-1)^4 + 4(-1)^3 - 2 = -1 - 4 - 2 = -7$

   $b = 4 \qquad f(4) = -\cancel{4^4} + \cancel{4 \cdot 4^3} - 2 = -2$

2. $f'(x) = 0 \qquad f'(x) = -4x^3 + 12x^2 = 0$

$$-4x^3 + 12x^2 = 0$$

$$-4x^2(x-3) = 0 \qquad \cancel{-4=0}$$

$$x^2 = 0 \qquad x = 0$$

$$x - 3 = 0 \qquad x = 3$$

$$f(0) = -\cancel{0^4} + \cancel{4 \cdot 0^3} - 2 = -2$$

$$f(3) = -3^4 + 4 \cdot 3^3 - 2 = -81 + 4(27) - 2 = 25$$

set each factor with $x$ equals to 0

$$\frac{-\cancel{4}x^2}{\cancel{-4}} = \frac{0}{-4}$$

$$x^2 = 0$$

$$x = 0$$

Answer: Absolute min of $f$ is $-7$

Absolute max of $f$ is $25$

Picture is above

CLASS WORK

Find the absolute max and the absolute min values of

$f(x) = \frac{x}{x^2+1}$ on $[0,2]$

closed interval

continuous in $(-\infty, \infty)$

$x^2+1=0$ never

$x^2=-1$

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 → absolute min |
| 2 | $2/5 = .4$ |
| 1 | $1/2 = .5$ |

absolute max

① $a=0$ $f(0) = \frac{0}{0^2+1} = \frac{0}{1} = 0$

$b=2$ $f(2) = \frac{2}{2^2+1} = \frac{2}{5}$

② $f'(x)=0$

$$f'(x) = \frac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2} = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

quotient rule

$$= \frac{-x^2+1}{(x^2+1)^2} = 0$$

NUMERATOR IS 0

$-x^2+1=0$ $1=x^2$ $x=\pm\sqrt{1}$ $x=\cancel{-1}$ → not in $[0,2]$

$x=1$

$f(1) = \frac{1}{1^2+1} = \frac{1}{2}$

Answer: 0 is the absolute min value of $f$

$\frac{1}{2}$ is the absolute max value of $f$

Good to graph to visualize

MEAN VALUE THEOREM

Let $f$ be a function that satisfies:

1) $f$ is continuous on $[a,b]$

2) $f$ is differentiable on $(a,b)$

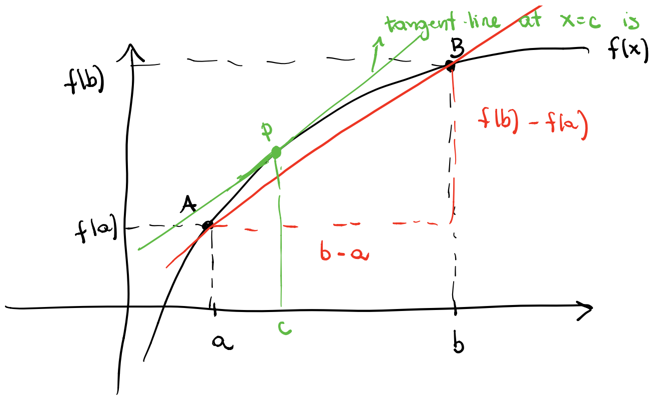
Then there is a number $c$ in $(a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

↓ slope of tangent line at $x=c$

↓ slope of line AB

There is a point P where the tangent line is parallel to the line AB




A = (a, f(a))

B = (b, f(b))

$$\frac{f(b) - f(a)}{b - a} = \frac{\text{RISE}}{\text{RUN}} = \text{slope of line AB}$$

AVERAGE SLOPE

MEAN SLOPE

<u>Ex</u> Verify that $f(x) = x^2$ satisfies the assumptions of the MVT on $[\overset{\text{"a"}}{0}, \overset{\text{"b"}}{2}]$. Find all numbers c that satisfy the conclusion of the MVT.

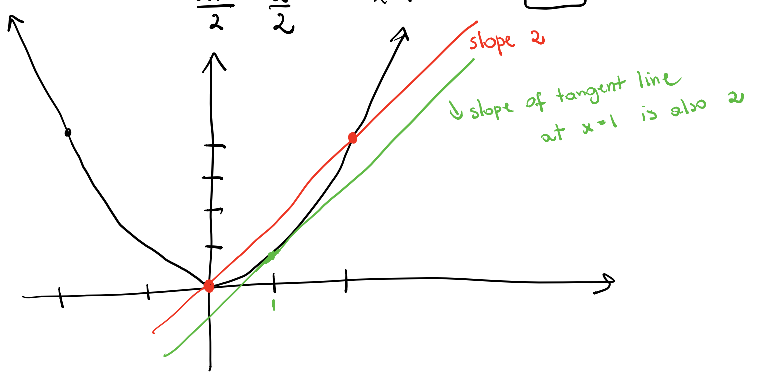
f(x) is a polynomial, so it is continuous and differentiable everywhere, so the MVT applies

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \frac{f(2) - f(0)}{2 - 0} = \frac{4 - 0}{2 - 0} = 2$$

$$f'(c) = \underbrace{\frac{f(2) - f(0)}{2 - 0}}_{\text{MEAN SLOPE}} = 2$$

$$f'(c) = 2 \qquad f'(x) = 2x$$

$$\frac{2x}{2} = \frac{2}{2} \qquad x = 1 \qquad \boxed{c = 1}$$




MAIN APPLICATION If f is a distance function, the MVT guarantees a time during the trip where your instantaneous speed is equal to the average speed.