§3.8 Implicit Differentiation

There are many ways to define a function. For example, a function can be defined explicitly or implicitly.

- Explicit functions: $y = f(x)$

  e.g. $y = x^2 - 1$

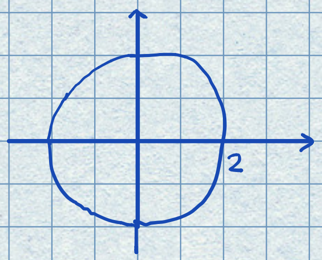
  $y = e^x \cdot \sin x - \ln x$

  in these functions, y is solved on the left hand side.

- Implicit functions: $F(x, y) = 0$

  e.g. $x^2 + y^2 = 4$

  this is a circle.



in these functions, y is not solved on one side and is mixed with the x in the equation.

Then how do we find y' or $\frac{dy}{dx}$?

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dt}(\sin t) = \cos t$ $\quad$ $\frac{d}{dy}(\sin y) = \cos y$

But what is $\frac{d}{dx}(\sin y)$?

we can use chain rule $\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y) \cdot \frac{dy}{dx}$

$= \cos y \cdot \frac{dy}{dx}$ OR $\cos y \cdot y'$

Let's practice more:

1. $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$ OR $2y \cdot y'$

2. $\frac{d}{dx}(\tan y) = \frac{d}{dy}(\tan y) \cdot \frac{dy}{dx} = \sec^2 y \cdot \frac{dy}{dx}$ OR $\sec^2 y \cdot y'$

3. $\frac{d}{dx}(3y^4) = \frac{d}{dy}(3y^4) \cdot \frac{dy}{dx} = 12y^3 \cdot \frac{dy}{dx}$ OR $12y^3 \cdot y'$

4. $\frac{d}{dx}(y) = \frac{d}{dy}(y) \cdot \frac{dy}{dx} = 1 \cdot \frac{dy}{dx} = \frac{dy}{dx}$ OR y'

Now we use the following example to show how to find $\frac{dy}{dx}$ for implicit functions $F(x,y)=0$

Example. Assuming that y is defined implicitly by

$$x^2 + y^2 = 25$$

find $\frac{dy}{dx}$

sol:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Example. Assuming that y is defined implicitly by

$$x^3 \sin y + y = 4x + 3$$

find $\frac{dy}{dx}$

sol:

$$\frac{d}{dx}(x^3 \sin y) + \frac{d}{dx}(y) = \frac{d}{dx}(4x) + \frac{d}{dx}(3)$$

$$\frac{d}{dx}(x^3) \cdot \sin y + x^3 \cdot \frac{d}{dx}(\sin y) + \frac{dy}{dx} = 4$$

$$3x^2 \cdot \sin y + x^3 \cdot \cos y \cdot \frac{dy}{dx} + \frac{dy}{dx} = 4$$

$$x^3 \cos y \cdot \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$

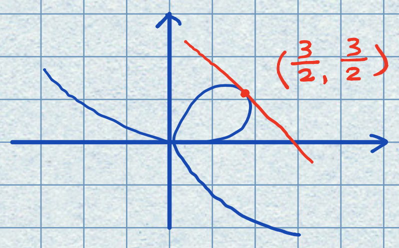
$$(x^3 \cos y + 1) \cdot \frac{dy}{dx} = 4 - 3x^2 \sin y$$

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

Example Find the equation of the tangent line to the graph of

$$y^3 + x^3 - 3xy = 0$$

at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$



sol: $$\frac{d}{dx}(y^3) + \frac{d}{dx}(x^3) - \frac{d}{dx}(3xy) = 0$$

$$3y^2 \cdot \frac{dy}{dx} + 3x^2 - \left(\frac{d}{dx}(3x)\cdot y + 3x \cdot \frac{d}{dx} y\right) = 0$$

$$3y^2 \cdot \frac{dy}{dx} + 3x^2 - \left(3y + 3x \cdot \frac{dy}{dx}\right) = 0$$

$$3y^2 \cdot \frac{dy}{dx} - 3x \cdot \frac{dy}{dx} = 3y - 3x^2$$

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$\Rightarrow m = \frac{3\left(\frac{3}{2}\right) - 3\left(\frac{3}{2}\right)^2}{3\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)} = -1$$

$$\Rightarrow \quad y = -1\left(x - \frac{3}{2}\right) + 3/2$$

$$y = -x + 3/2 + 3/2$$

$$\boxed{y = -x + 3}$$

Example. Find a second derivative $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$

sol: First find $\frac{dy}{dx}$:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

then find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x}{y}\right)$$ derivative of the derivative

$$= \frac{(-x)' \cdot y + (-x) \cdot y'}{y^2}$$

$$= \frac{-y - xy'}{y^2} = \frac{-y - x \cdot \left(\frac{-x}{y}\right)}{y^2}$$

$$= \frac{-y^2 + x^2}{y^3}$$