Class #15 - Monday, October 28
Section 4.7: Counting Principles (Permutations & Combinations)

Definition: A **permutation** of a set of objects is an arrangement of the objects in a particular order. For example, the permutations of the set $\{a, b, c\}$ are:

$$abc, acb, bac, bca, cab, cba$$

Note that there are $3! = 3 * 2 * 1 = 6$ different permutations of the 3 objects $\{a, b, c\}$. In general, there are $n!$ different permutations of n distinct objects. (Why?)

(Recall: n factorial: $n! = n * (n-1) * (n-2) * \ldots * 1$)

Example 1: Suppose there are 5 people running a race. In how many different orders can the runners finish the race?

In many applications, instead of counting the number of permutations of all n objects in a given set, we just want count the number of permutations of some fixed number r of objects taken from the set (where $r < n$).

Example 2: Consider again the 5-person race, but now we just want to figure out how many different 1st-2nd-3rd place results there can be; i.e., we want to figure out the number of permutations of the 8 people taken 3 at a time.

Permutations of n objects taken k at a time: The number of "permutations of n distinct objects taken k at a time" (or "k-permuations of n") is

$$P(n, k) = P_{n,k} = \frac{n!}{(n-k)!} = n * (n-1) * (n-2) * \ldots * (n-k+1)$$

(Note that the product on the RHS has k factors, starting with k and decreasing by 1. It should make sense why this product gives you the number of such permutations. The formula using factorials is just a convenient way to express that product.)

Spreadsheet command: `=permut(n,r)`

Example 3: To see the $P(5, 3) = 5 * 4 * 3 = 60$ different 3-permutations of $\{A, B, C, D, E\}$ in a Google spreadsheet click here. (These permtuations were generated using the website `http://www.dcode.fr/partial-k-permutations`.)

Example 4:

(i) Forty-three race cars started the 2010 Daytona 500. How many ways could the cars have finished first, second, and third?

(ii) The board of directors of a company has 12 members. One member is the president, another is the vice president, another is the secretary, and another is the treasurer. How many ways can these positions be assigned?

(iii) Suppose a lottery consists of 5 different numbers chosen from 1-59. To win the lottery, you must have guessed the 5 numbers in order. What is the probability of winning the lottery?

Combinations

Now suppose we need to choose r objects from a set of n objects, but now the *order does not matter*. These are called **combinations** (of n objects taken r at a time).

Example 5: Suppose five friends–Alice, Bob, Charlie, David and Edgar ($\{A, B, C, D, E\}$ for short)–have only 3 tickets to a concert. How many different ways could they distribute the tickets among them? In other words, how many combinations are there of the $n = 5$ people taken $r = 3$ at a time?

Note there are $P_{5,3} = 5 * 4 * 3 = 60$ different permutations. Here are the 12 of them beginning with A:

$$ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, \ldots$$

But certain groups of these permutations correspond to the same combination. For example, take the combination $\{A, B, D\}$. In the complete list of 60 permutations, there are $3! = 6$ different permutations which correspond to this same combination: $ABD, ADB, BAD, BDA, DAB, DBA$.

This will be true of any such combination of size 3: there will be 3! corresponding permutations. (Try listing them for a different combination, say for $\{C, D, E\}$.)

So to get the number of combinations, we *divide* the number of permutations by 3!:

$$\binom{5}{3} = C_{5,3} = C(5,3) = \frac{5 * 4 * 3}{3 * 2 * 1} = \frac{60}{6} = 10$$

This is the reasoning behind the general formula:

Combinations of n objects taken r at a time: The number of "combinations of n distinct objects taken r at a time" is

$$\binom{n}{r} = = C_{n,r} = C(n,r) = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!} = \frac{n * (n-1) * (n-2) * \ldots * (n-r+1)}{r!}$$

Spreadsheet command: `=combin(n,r)`