An Introduction to Quantum Algorithms for MAT 2440

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Goals:

1. The "intellectual core" of computer science - review of computational complexity.
2. Classical key exchange protocols and RSA cryptosytems
3. Moving from the classical to the quantum
4. What is a quantum bit (qubit)?
5. Advantages of quantum computation vs. classical computation
6. Reversible logic Gates, quantum logic gates
7. Breaking RSA! - Shor's algorithm for the quantum factorization of integers
8. What else can we do? Grover's search procedure.
9. Conclusions and where are we headed?
10. References

Computational Complexity - Hard, Harder, Hardest

Efficient Algorithms

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Problems that cannot be solved using an algorithm with worst-case polynomial time complexity are called **intractable**.

Some important classes of problems

P and NP

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NP-complete

There is a class of problems with the property that if any of these problems can be solved by a an efficient algorithm, then all problems in the class can be solved by an efficient algorithm. They are in essence the "same" problem!!!

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4. Every known algorithm for these problems will take an amount of time that increases **exponentially** with the problem size.
5. These are all the "same" in that an efficient algorithm for solving one of them will imply an efficient algorithm for solving all of them.

$P = NP$?

The million dollar question (**literally**)

An efficient algorithm for an NP-**complete** problem would mean that computer scientists' present picture of the classes P, NP and NP-**complete** was utterly wrong!!! It would mean that $P = NP$!

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NOTE: The factoring problem is neither known nor believed to be NP-**complete**.

RSA cryptosystems

In the RSA (Rivest, Shamir and Adleman)cryptosystem, each individual has an encryption key (n, e) where $n = pq$ the modulus is the product of two large primes p and q, (say with 200 digits each), and an exponent e that is relatively prime to $(p-1)(q-1)$

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In order to *decrypt* a message sent in this scheme, once must be able to find the decyrption key d which is the inverse of e, $\mod (p-1)(q-1)$. Without knowing p and q, and only knowing n (with possibly 400 digits), this system is difficult to break!

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Quantum effects are beginning to interfere with electronic devices as they are made smaller and smaller.

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In 1985 David Deutsch defined quantum Turing machines, a theoretical model for quantum computing.

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- A qubit can be a photon in a vertically polarized state ("1") or a horizontally polarized state ("0").

Where do qubits "live"?

Disclaimer: These mathematical objects will become clearer to you once you have taken a class in LINEAR ALGEBRA! You are not expected to understand these statements YET.

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We can generalize the concept of qubits to quantum registers. A state of a quantum register of size n is a *tensor product* of n qubits and can be written as

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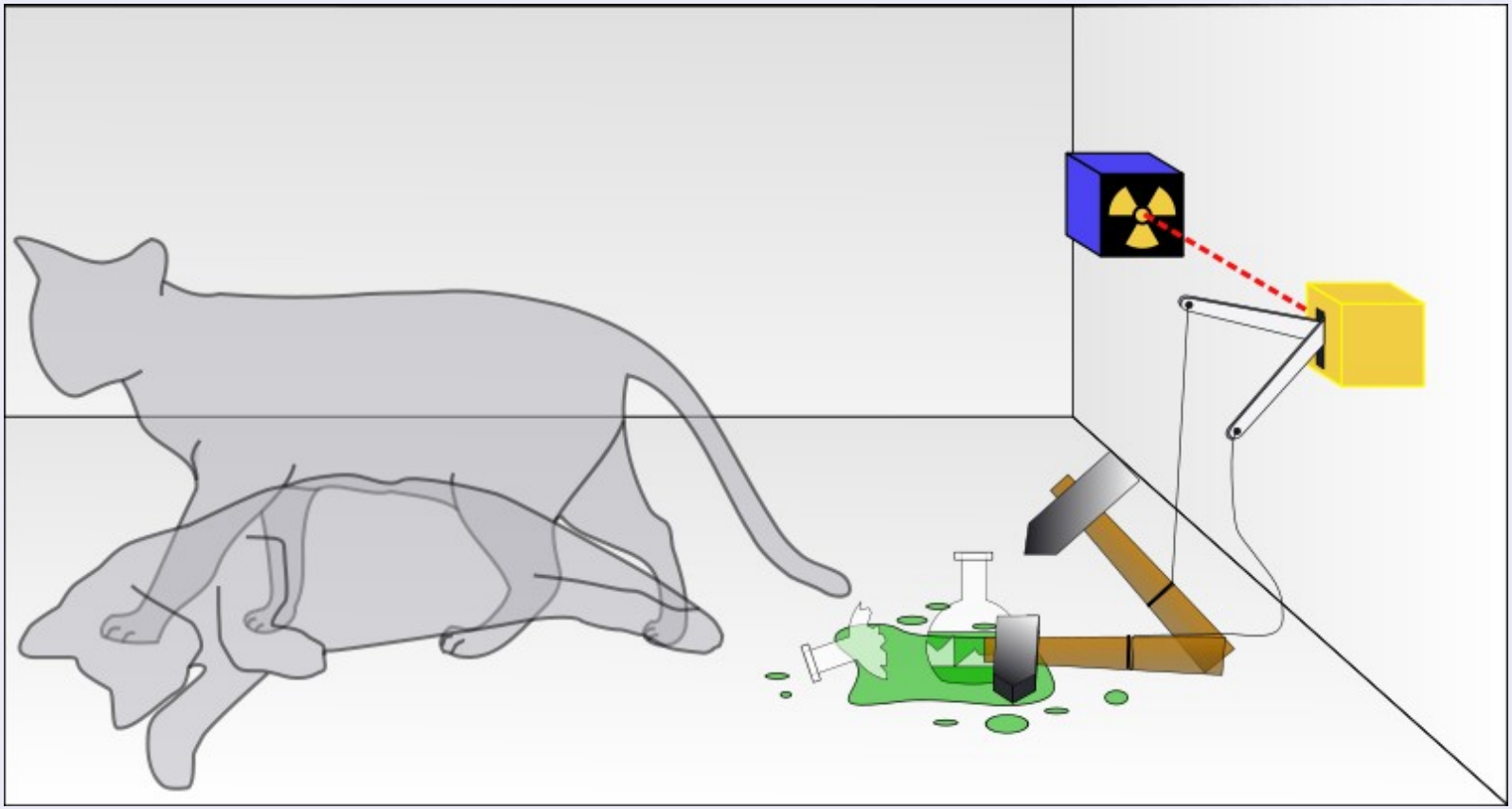
The size of the computational state space of a quantum register is exponential in the physical size of the system.

How can a cat be both dead and alive?



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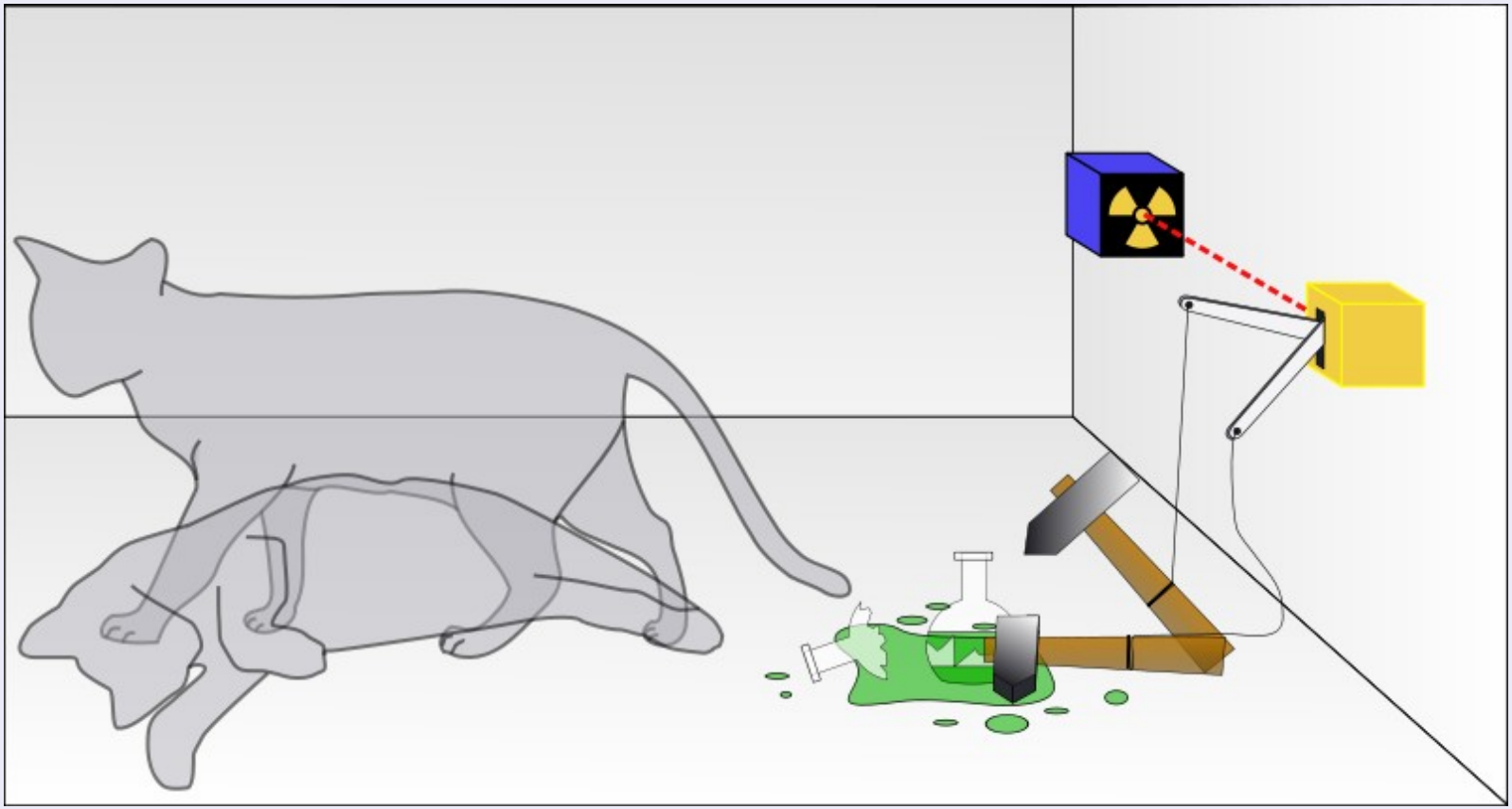

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- `https://www.youtube.com/watch?v=IrbJYsep45E`

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- In 1973 Bennett found that any computation can be performed using only reversible steps.

Example

The following is the truth table for the three-bit CONTROL-CONTROL-NOT gate (or CCN-gate).

a_i	b_i	c_i
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- Quantum logic gates generally act on a superposition of digital states. They can be represented by operators (matrices).
- Unitary matrices represent the time evolution of quantum mechanical systems.
- So quantum logic gates can be represented by unitary matrices.

The Quantum N-gate

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

transforms $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$.

Quantum Circuits

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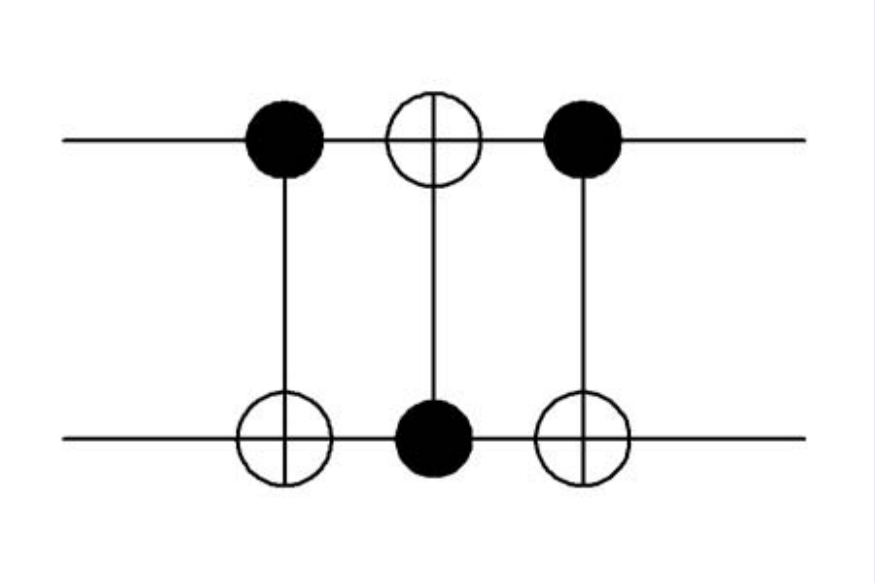

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- For instance, RSA-768, the largest number to be factored to date, had 232 decimal digits and was factored over multiple years ending in 2009, using the equivalent of almost 2000 years of computing on a single 2.2 GHz AMD Opteron processor with 2GB RAM.2!

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- On a classical computer the best known method for factorization of a number with 300 digits takes $5 \cdot 10^{24}$ steps or with terahertz speed **150,000 years**!

Shor's Algorithm

Shor's algorithm hinges on being able to quickly compute the period of the following periodic function (using a quantum algorithm):

$$f(x) = y^x \mod (n)$$

for $x = 0, 1, 2, \ldots$. Where $n = pq$, p and q prime factors.

To begin, select y randomly so that $1 < y < n$ and $(y, n) = 1$. Next, find the period T of the function $f(x) = y^x \mod (n)$. Compute $z = y^{T/2}$. Lastly, to find factors of n, compute $(z + 1, n)$ and $(z - 1, n)$ (can use the Euclidean Algorithm.

Shor's Algorithm: An example

Factoring $n = 30$ using Shor's Algorithm

- Given $n = 30$, choose a y, so that $1 < y < 30$ and $(y, 30) = 1$. I will use $y = 11$ (but I could have chosen: 19, 29, 7, 13 or 23).
- Compute values of $f(x)$ to find the period of $f(x)$ (by hand):

$$f(0) = 11^0 \mod 30 = 1$$

$$f(1) = 11^1 \mod 30 = 11$$

$$f(2) = 11^2 \mod 30 = 1$$

$$f(3) = 11^3 \mod 30 = 11$$

$$f(4) = 11^2 \mod 30 = 1 \ldots$$

- T is obvious 2. Now compute $z = y^{T/2} = 11^{2/2} = 11^1 = 11$.
- Finally factors of $n = 30$ are $(z + 1, 30) = (12, 30) = 6$ and $(z - 1, 30) = (10, 30) = 10$. Both 6 and 10 are factors of 30.

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- Also note, that for $y = 11, 19, and 29,\ T = 2$ and for $y = 7, 13, and 23,\ T = 4$.

Your turn to try!

Use Shor's algorithm to factor $n = 21$. You may use our Python code for fast modular exponentiation to help you find the period of $f(x)$ and the code for computing the gcd if you would like.
`https://trinket.io/python/653108cbc9`
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The Quantum Search Algorithm

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Assume $N = 2^n$ so the index is stored in n bits and that the search problem has one solution.

Define a function $f(x)$, such that for $x \in \{0, 1, \ldots, N-1\}$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not a solution} \\ 1 & \text{if } x \text{ is a solution.} \end{cases}$$

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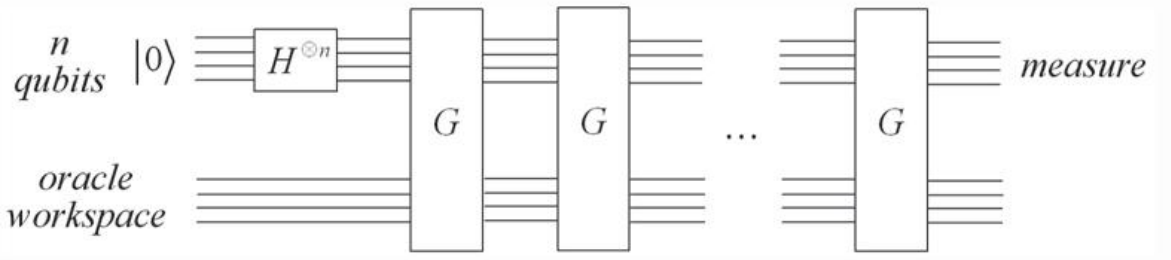
The oracle, O, is a unitary operator defined on the computational basis $|0\rangle$, $|1\rangle$.
The action of the oracle is

$$O|x\rangle|q\rangle = |x\rangle|q \oplus f(x)\rangle$$

where x is the index register and $|q\rangle$ is the oracle qubit.



In order to achieve the correct solution with probability near 1, one must apply the Grover iteration $O\left(\sqrt{N}\right)$ times if there is a single solution.

Boyer, Brassard, Høyer and Tapp showed that if it is known in advance that there are M solutions to the search problem, one must apply the Grover iteration $O\left(\sqrt{\frac{N}{M}}\right)$ times.

Geometric Visualization

One can view the Grover iteration as a rotation in the two dimensional Hilbert space spanned by the initial vector $|\psi\rangle$ and the state that consists of the superposition of the solutions to the search problem.

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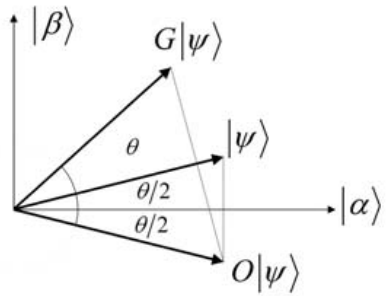
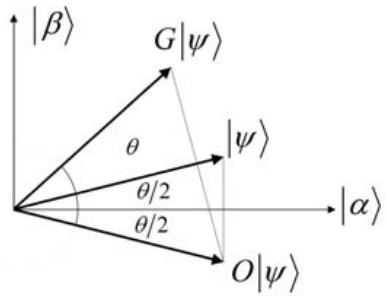


Figure: The action of a single Grover iteration G.

- No matter how many times G is applied to $|\psi\rangle$, the vector remains in the plane spanned by $|\alpha\rangle$ and $|\beta\rangle$.



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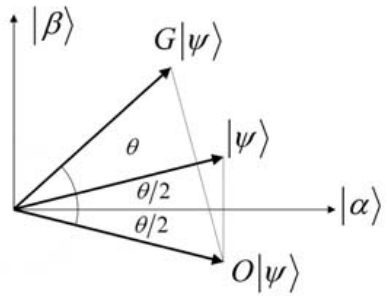

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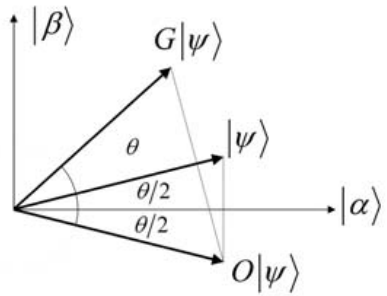

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- Each application of G rotates the vector $|\psi\rangle$ closer to alignment with the vector $|\beta\rangle$.
- When this occurs, a measurement in the computational basis produces one of the outcomes superimposed on $|\beta\rangle$ with high probability.
- This is a solution to the search problem.

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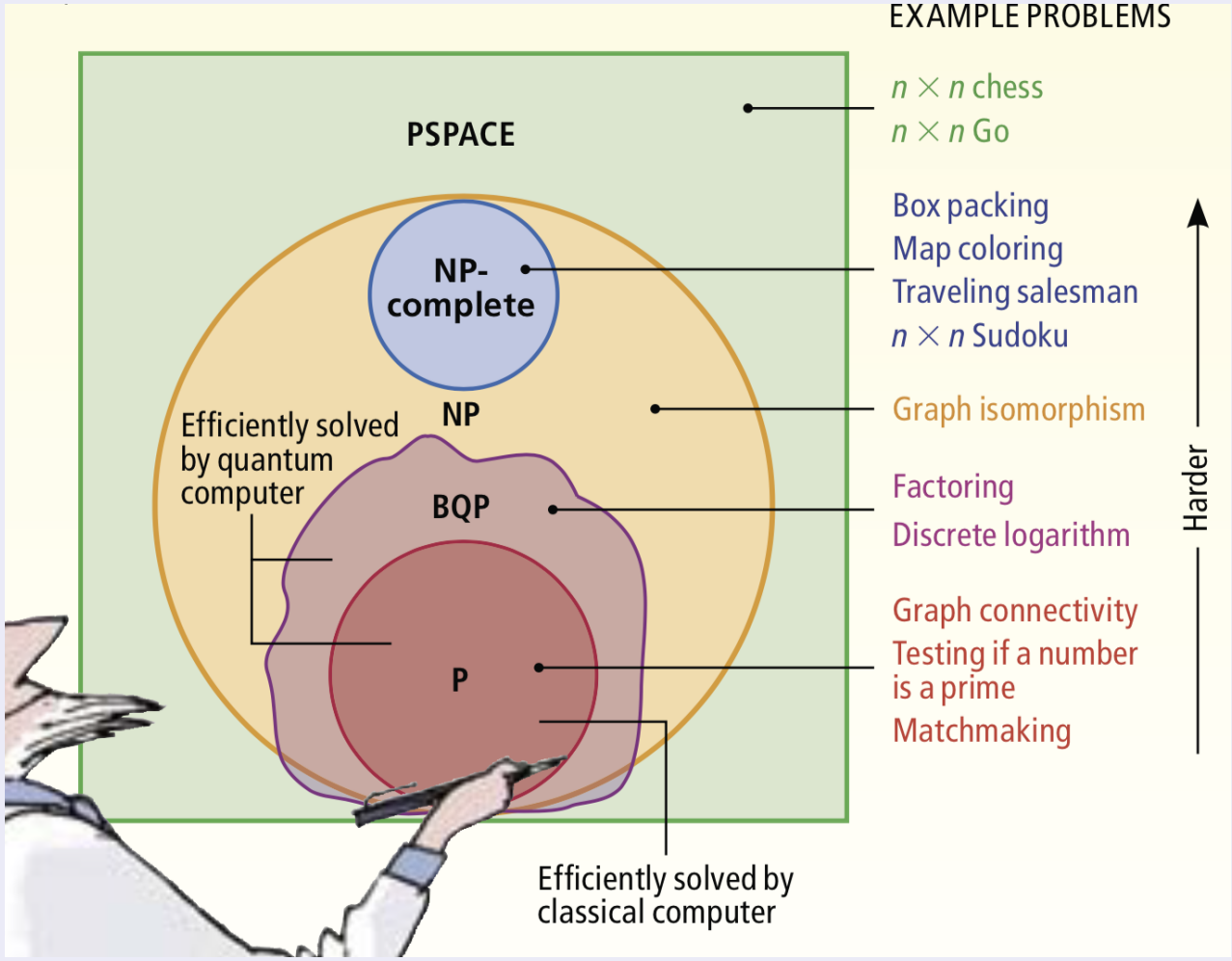
- A goal is to determine for which problems quantum computers are faster than classical computers.
- Two important quantum complexity classes are BQP and QMA which are the bounded-error quantum analogues of P and NP.
- Goal: Find out where these classes lie with respect to classical complexity classes such as P, NP, PP (problems solvable by probabilistic Turning machine in poly-time), PSPACE (problems that can be solved by a Turing machine using a poly-space) and other complexity classes.

Our best guess:



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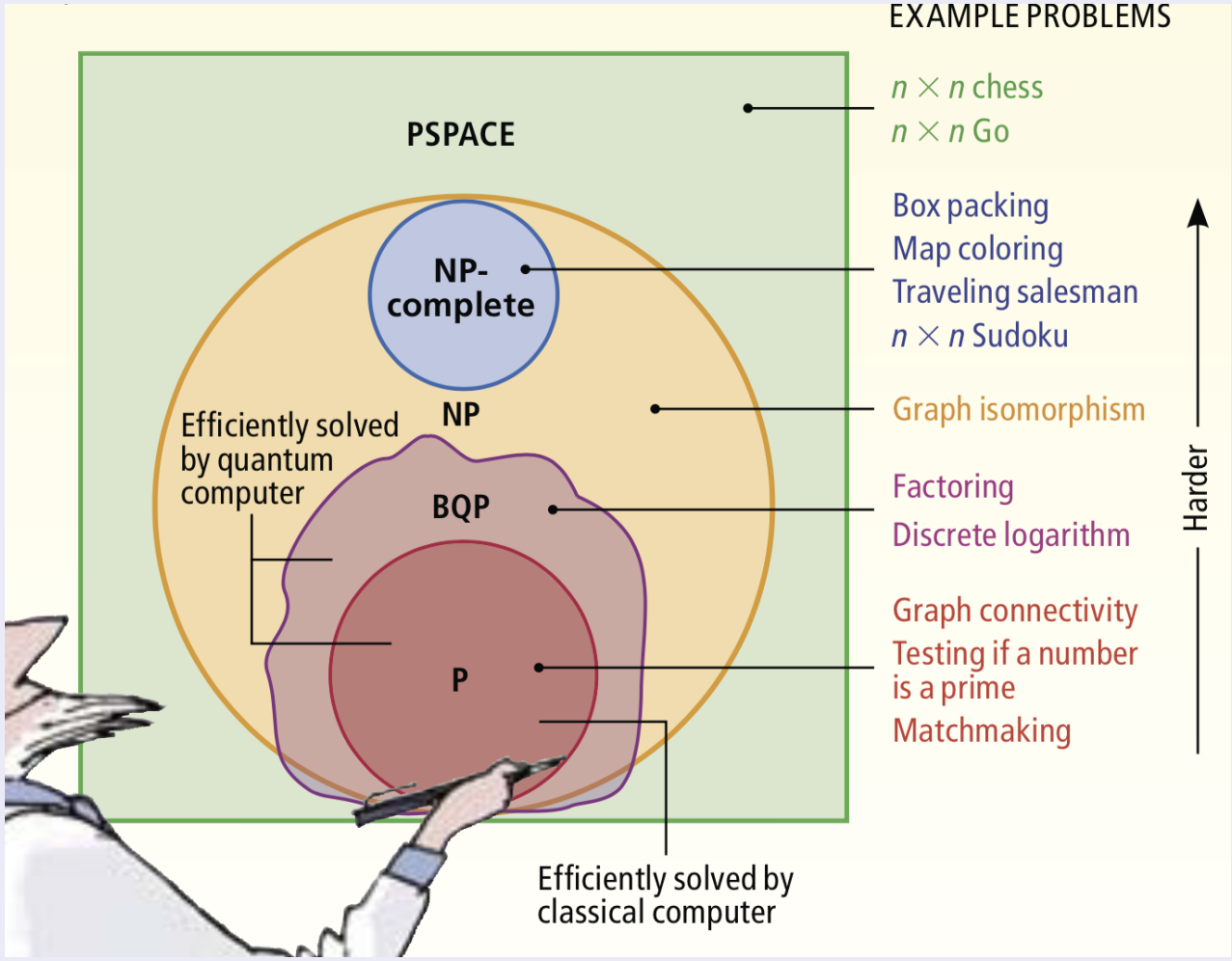

The diagram depicts the class of problems a quantum computer would solve efficiently, BQP ("bounded error, probabilistic, polynomial time"), might relate to other fundamental classes of computational problems.

Timeline:

- **1998**
  - First working 3-qubit Nuclear Magnetic Resonance (NMR) computer.
  - First execution of Grover's algorithm on an NMR computer.
- **2000**
  - First working 5-qubit NMR computer demonstrated at the Technical University of Munich.
  - First execution of order finding (part of Shor's algorithm) at IBM's Almaden Research Center and Stanford University.
  - First working 7-qubit NMR computer demonstrated at the Los Alamos National Laboratory.
- **2001**
  - First execution of Shor's algorithm at IBM's Almaden Research Center and Stanford University. The number 15 was factored.

- **2006**
  - First 12 qubit quantum computer benchmarked at the Institute for Quantum Computing (IQC) and PI in Waterloo.
- **2008**
  - D-Wave Systems claims to have working 28-qubit quantum computer.
- **2009**
  - Google collaborates with D-Wave Systems on image search technology using quantum computing.

- **2010**
  - D-Wave claims to have developed quantum annealing and introduces their product called D-Wave One. The company claims this is the first commercially available quantum computer.
  - Practical error rates achieved.
- **2012**
  - Physicists Create a Working Transistor From a Single Atom.
  - D-Wave claims a quantum computation using 84 qubits.
- **2013**
  - Documents leaked by Edward Snowden revealed that the NSA worked to "Insert vulnerabilities into commercial encryption systems, IT systems, networks, and endpoint communications devices used by targets" as part of the Bullrun program.

- **2014**
  - Documents leaked by Edward Snowden also confirm the Penetrating Hard Targets project, by which the NSA seeks to develop a quantum computing capability for cryptography purposes.
- **2015**
  - Quantum error detection code using a square lattice of four superconducting qubits.
  - D-Wave Systems Inc. announced on 22 June that it had broken the 1000 qubit barrier.
- **2016**
  - Google, using an array of 9 superconducting qubits developed by the Martinis group and UCSB, accurately simulates a hydrogen molecule.

- **2017**
  - D-Wave Systems Inc. announced on 24 January general commercial availability of the D-Wave 2000Q quantum annealer, with 2000 qubits.
  - Working blueprint for a microwave trapped ion quantum computer published in Science Advances by international collaborators.
  - IBM unveils 17-qubit quantum computerand a better way of benchmarking it.

- **2018**
  - In late 2017 and early 2018 IBM, Intel, and Google each reported testing quantum processors containing 50, 49, and 72 qubits, respectively, all realized using superconducting circuits.
  - In July 2018, a team led by the University of Sydney has achieved the world's first multi-qubit demonstration of a quantum chemistry calculation performed on a system of trapped ions, one of the leading hardware platforms in the race to develop a universal quantum computer.

Thank you!

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- An article, "The Limits of Quantum Computers" https://www.cs.virginia.edu/~robins/The_Limits_of_Quantum_Computers.pdf
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- Another perspective: https://www.worldscientific.com/worldscibooks/10.1142/3808