MAT 2440 Assignment #2 - The Lamplighter Group L_2

1 L_2 as a dynamical system

We take our definition of dynamical system to be an "object" along with a specific set of modifications that can be performed (dynamically) upon this object. In this case, the object is a bi-infinite straight road with a lamp post at every street corner. There are two possible types of modifications: the lamplighter can walk any distance in either direction from a starting point and the lamplighter can turn the lamps "on" or "off." At any given moment the lamplighter is at a particular lamp post and a finite number of lamps are illuminated while the rest are not. We refer to such a moment, or configuration, as a "state" of the road (not to be confused with the "state" of an automaton). Any time the configuration changes, the road is in a new state. The road's state is changed over time by the lamplighter either walking to a different lamp post or turning lamps on or off (or both).

In Figure 1, the bi-infinite road is represented by a number line; the lamps are indexed by the integers. Lamps that are on are indicated by stars; lamps that are off by circles. The position of the lamplighter is indicated by an arrow pointing to an integer. The current state of the road is called the *lampstand*.

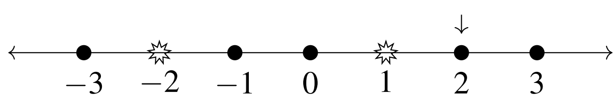


Figure 1: *A lampstand where two lamps are illuminated and the lamplighter stands at 2.*

Let us call the set of all possible lampstands $\mathscr{L}$. Now that we have a visual image, we can formalize the dynamics of changing a lampstand by specifying distinct tasks which the lamplighter can perform on any element of $\mathscr{L}$.

1. Move right to the next lamp.

2. Move left to the next lamp.

3. Switch the current lamp's status (from on to off or off to on).

4. Do nothing.

For any reconfiguration, the lamplighter performs only finitely many tasks. These tasks can be interpreted as functions τ, σ and I, whose domain and range are $\mathscr{L}$. Given a lampstand $l \in \mathscr{L}$, $\tau(l)$ is the result of performing the first task on l, $\sigma(l)$ is the result of performing the third task on l and $I(l)$ the result of performing the fourth task on l.

Proposition 1. σ is bijective.

Proof. To see that σ is onto, let l_1 be any lampstand in $\mathscr{L}$, and suppose that the lamplighter stands at lamp k. Define l_0 as the lampstand whose lamplighter stands at lamp k and whose lamps are in the same configuration as those in l_1, **except** for lamp k. If k is on in l_1, it is off in l_0; if it is off in l_1, it is on in l_0. Then $\sigma(l_0) = l_1$.

To see that σ is one-to-one, suppose that $\sigma(l_0) = \sigma(l_0') = l_1$, with the lamplighter in l_1 standing at lamp k. Since σ does not cause the lamplighter to move, the only effect it has on a lampstand is to switch the status of the current lamp. Whatever the status of lamp k is in l_1, it must be in the opposite state in both l_0 and l_0'. All other attributes of both l_0 and l_0' must match the other attributes of l_1; hence, $l_0 = l_0'$.

□ □

The reader will prove that τ is also bijective in Exercise 2 at the end of this chapter. Hence, both σ and τ have inverses. $\tau^{-1}(l)$ is the result of performing the second task on l. Note that σ is its own inverse. Thus $\sigma^2 = 1$.

If we let the lamplighter stand at 0 with all the lamps turned off, this configuration is called the *empty lampstand* and is denoted e. See Figure 2.

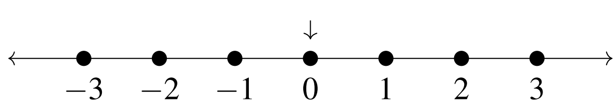


Figure 2: *The empty lampstand e*

Example 1. Consider the lampstand l_1 in Figure 3.

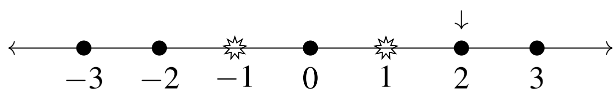


Figure 3: *The lampstand l_1*

Starting with the empty lampstand e, we can apply a composition of functions τ, τ^{-1}, σ and I to achieve l_1. For instance the composition $\tau\sigma\tau\tau\sigma\tau^{-1}$ (or $\tau \circ \sigma \circ \tau \circ \tau \circ \sigma \circ \tau^{-1}$) applied to e yields the lampstand configuration l_1. In keeping with standard function notation, the order of the composition is such that τ^{-1} is applied to e first and so on, reading from right to left. Figure 4 shows the details of the transformation from e to l_1.

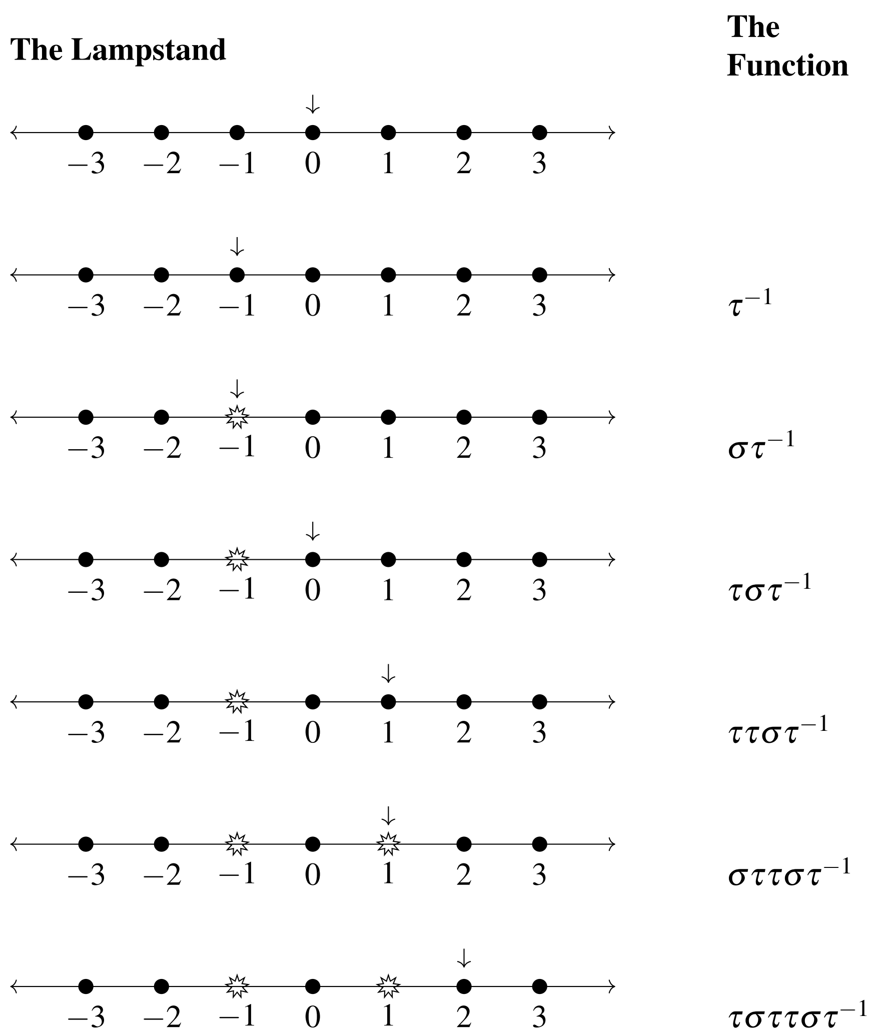


Figure 4: *A sequence of lampstands from the empty lampstand to* l_1 $\diamond$

To get the same lampstand l_1 (Figures 3 and 4), we could easily have applied a different function composition to e, for instance

$$\tau I \tau\tau I \sigma \tau^{-1} \tau^{-1} \sigma \tau.$$

For that matter, pick any $l \in \mathscr{L}$ as input. These two different-looking functions always have the same output.

$$\tau\sigma\tau\tau\sigma\tau^{-1}(l) = \tau I\tau\tau I\sigma\tau^{-1}\tau^{-1}\sigma\tau(l).$$

It doesn't matter that there are different function compositions representing the same lampstand, since two functions are defined to be the same function as long as the domains are the same and the outputs are the same. However, some function compositions are clearly "shorter" than others. Here "shorter" refers to the number of tasks in the function composition. This begs the question, is there a "shortest" function composition for a given lampstand configuration? You will explore this in the Exercise 4 below.

2 Assignment #2

This assignment is due on 3/12/2019 at 10 am - at the beginning of our class period. You may submit it electronically as a pdf document or as a hard copy. Assignments late by 1 day will be penalized by 25%, 2 days late 50%, 3 days late 75% and any later they will no longer be accepted. You must submit your OWN work, you may not submit a group report.

Exercise 1. Check that the lampstand l_2 shown in Figure 5 can be arrived at by starting with the empty lampstand and applying this composition of functions - show/draw all steps: [20 points]

$$\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma.$$

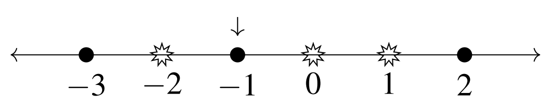
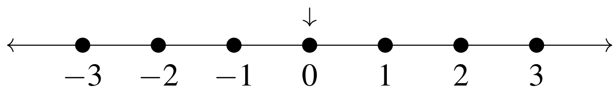
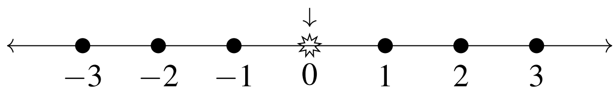
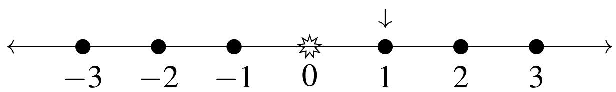
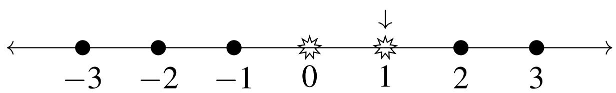
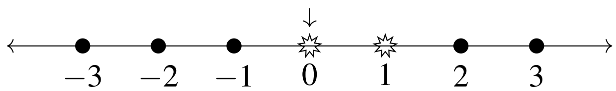
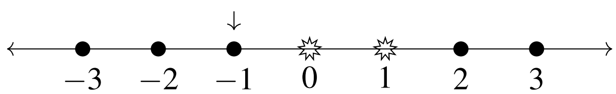
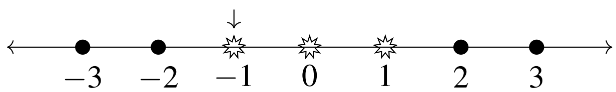
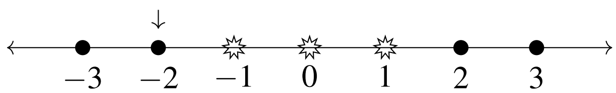
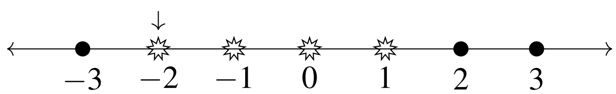
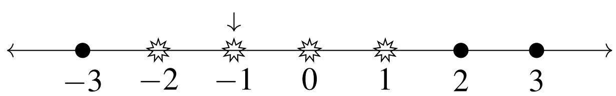
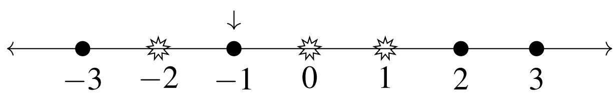


Figure 5: *The lampstand l_2*

The Lampstand composition of $\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$	**The Function**
	
	σ
	$\tau\sigma$
	$\sigma\tau\sigma$
	$\tau^{-1}\sigma\tau\sigma$
	$\tau^{-1}\tau^{-1}\sigma\tau\sigma$
	$\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$
	$\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$
	$\sigma\,\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$
	$\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$
	$\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$

Exercise 2. Prove that τ is bijective. Please be sure to write out your reasoning and show all of your steps. [30 points]

Proof. The goal is to prove that τ is bijective. To see that τ is onto, let l_1 be any lampstand in $\mathscr{L}$, and suppose that the lamplighter stands at lamp k. Define l_0 as the lampstand whose lamps are in the same configuration as those in l_1, **and** the lamplighter is standing at lamp $k-1$. Then $\tau(l_0) = l_1$.

To see that τ is one-to-one, suppose that $\tau(l_0) = \tau(l'_0) = l_1$. Since τ does not turn lamps "on" or "off," the only effect it has on a lampstand is to move the lamplighter to the right one unit. Wherever the lamplighter is positioned in l_1 (at k), that position must be to the right of where the lamplighter is in l_0 or l'_0. All other attributes of both l_0 and l'_0 must match the other attributes of l_1; hence, $l_0 = l'_0$.

□ □

Exercise 3. Draw your own lampstand configuration and write down its function composition (consisting of τ, τ^{-1} and σ) and **share it** with one of your team members. You must also include: **the drawing of the lampstand** and **its function composition** here with this assignment in order to receive credit for this part. [10 points]

Exercise 4.

a. Consider the lampstand shared with you by your fellow team member. Write down **two more** distinct function compositions (consisting of τ, τ^{-1} and σ) besides the one shared with you to represent your fellow team member's lampstand. [10 points]

b. Of your three different function compositions (from part a.) representing the lampstand shared with you, which is the "shortest"? How many function compositions of τ, τ^{-1} and σ does it have? [5 points]

c. What is the minimum number of function compositions necessary to express the lampstand shared with you? [5 points]

d. Describe a general method for finding the shortest function composition to represent *any* lampstand l where the lamplighter stands to the left of 0, with finitely many positive and negative lamps lit. [10 points]

e. How would you modify your method if the position of the lamplighter was to the right of 0? [10 points]

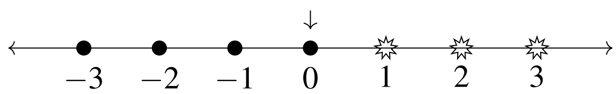


Figure 6: *The lampstand representation for* g_1

A discussion on minimizing the length of function compositions ("words") representing lampstands, note that in this discussion, the lamplighter's name is Hanna:

In order to calculate word length, we will focus on the reasoning rather than giving a proof. There are two different types of lampstands to consider:

Case 1) Hanna is standing at 0.

Case 2) Hanna is standing somewhere other than 0.

To explore Case (1), let us continue to consider the lampstand configuration given by g_1 where Hanna is standing at 0 and the lamps at 1, 2 and 3 are lit (see Figure 6).

One way this configuration can be achieved efficiently is for Hanna to light the lamps at 1, 2 and 3 as she travels out to 3 and then moves back to stand at 0. The function composition representing g_1 in this case is

$$\tau^{-3}\sigma\tau\sigma\tau\sigma\tau,$$

for a total of nine functions being composed; i.e., $c(\tau^{-3}\sigma\tau\sigma\tau\sigma\tau) = 9$. Similarly, Hanna could move to 3 first and then light the lamps at 3, 2 and 1 on her way back to stand at 0. In this case, g_1 is represented by

$$\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^{3}$$

which also uses nine functions. Can you use fewer tasks? Try! It turns out that 9 is the minimum number of function compositions required to represent g_1 (using τ, τ^{-1} and σ), so $|g_1| = 9$. This makes sense intuitively; we have eliminated any unproductive moving back and forth of Hanna. To make the order of lighting the lamps consistent, from now on Hanna will light the necessary lamps as she travels outward from the origin.

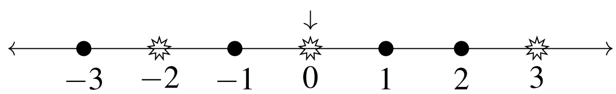


Figure 7: *The lampstand representation for g_2*

Example 2. Let $g_2 = \tau^{-4}\sigma\tau\sigma\tau^{-1}\sigma\tau^{6}\sigma\tau^{-2}\sigma$. Following Hanna's moves (try it!), we get the lampstand shown in Figure 7.

Since Hanna's final position is at the origin, she should avoid crossing back and forth over 0 as she lights the lamps. Suppose that from 0 she moves to the **right** first, lighting lamps in ascending order. Next she moves **left** (to 0 and beyond), lighting nonpositive lamps in descending order, before moving to her final position at 0. When using this method to create a lampstand configuration, the resulting word representation is referred to as the *right-first* representation associated with word w, denoted $rf(w)$. Hanna could also have moved left first, lighting lamps in descending order starting at 0, and then right, lighting the lamps beyond 0 in ascending order before moving into her final position. This result is referred to as the *left-first* representation associated with word w, denoted $lf(w)$.

When trying to find a representation of an element of L_2 that contains the minimum number of generators, the right-first and left-first representations are our candidates. Which is "shorter," the right-left or left-right representation? In this case, they are the same, since

$$rf(g_2) = \tau^2\sigma\tau^{-2}\sigma\tau^{-3}\sigma\tau^3 \text{ and } lf(g_2) = \tau^{-3}\sigma\tau^5\sigma\tau^{-2}\sigma.$$

Both representations use 13 generators to accomplish the desired lampstand configuration. Thus, $|g_2| = 13$.

Whenever Hanna's final position is at the origin, it will be the case that both $rf(w)$ and $lf(w)$ will use the minimum number of generators for a given lampstand.
◇

Case (2) is more complicated: Hanna's final position is somewhere other than 0, and a possible mix of positive and negative lamps are lit.

Example 3. Consider the element g_3 (see Figure 8).

Again we will look at the right-first and left-first representations of g_3 in order to determine the most efficient way of accomplishing this configuration (i.e. minimizing the instances of τ,τ^{-1} and σ). For the right-first representation ($rf(w)$),

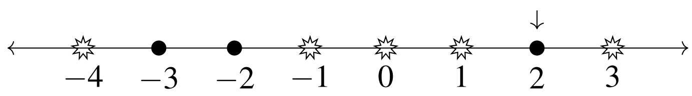


Figure 8: *The lampstand representation for g_3*

Hanna begins at 0 and moves to the **right** first, lighting lamps in ascending order, then moves left to -4, lighting nonpositive lamps in descending order, before moving to her final position at 2. For the left-first representation ($lf(w)$), Hanna begins at 0 and moves to the left, lighting lamps in descending order starting at 0 and then right, lighting lamps beyond 0 in ascending order before moving to her final position.

There are 20 occurrences of generators in $rf(g_3)$ since

$$rf(g_3) = \tau^6\sigma\tau^{-2}\sigma\tau^{-1}\sigma\tau^{-3}\sigma\tau^2\sigma\tau$$

and 17 occurrences in $lf(g_3)$ since

$$lf(g_3) = \tau^{-1}\sigma\tau^2\sigma\tau^5\sigma\tau^{-3}\sigma\tau^{-1}\sigma.$$

$c(rf(g_3)) = 20$ and $c(lf(g_3)) = 17$; thus, $|g_3| = 17$.
◇

What made the left-first word more efficient? If you think for a moment you will realize that the left-first word has a smaller count than the right-first word due to the fact that Hanna finished her work by standing to the **right** of 0. Once all the lamps were lit, she didn't have to travel past 0 a second time to reach her final position.

To summarize, when looking for the length of a word w in L_2, where the lamplighter's position is at lamp l,

$$|w| = \begin{cases} c(rf(w)) & \text{if } l < 0 \\ c(lf(w)) & \text{if } l > 0 \\ c(rf(w)) = c(lf(w)) & \text{if } l = 0. \end{cases}$$

Minimizing the number of generators to represent a word consists of having the lamplighter head away from zero in the opposite direction from her final position. If her final position is at zero, she can head in either direction.

Given a word $w \in L_2$, we now know how to find its length, that is, if we are willing to take the time to compute one or the other of $c(rf(w))$ and $c(lf(w))$. For many purposes – this is sufficient.