Sample Exam # 2 - Solutions

1.

a) pf | n = odd int = $2k+1$ for $k \in \mathbb{Z}$

$$2k+1 = \underbrace{(k+1)^2 - k^2}_{\text{difference of 2 squares}} = k^2 + 2k + 1 - k^2$$

$$= 2k+1 = n \quad //$$

difference of 2 squares

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b) pf use contrapositive:

$\neg(n \text{ even}) \rightarrow \neg(n^3+5 \text{ odd})$

i.e. n odd $\rightarrow n^3+5$ even

Let $n = 2k+1$ $k \in \mathbb{Z}$

then $n^3 + 5 = (2k+1)^2(2k+1) + 5$

$= (4k^2 + 4k + 1)(2k+1) + 5$

$= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 + 5$

$= 8k^3 + 12k^2 + 6k + 6$

$= 2(\underbrace{4k^3 + 6k^2 + 3k + 3}_{\text{some integer call it } p})$

some integer call it p

$= 2p$

$\Rightarrow n^3+5$ is even. //

1.

c) $\longrightarrow$

pf/ $n \in \mathbb{Z}^+$ and n odd $\rightarrow$ $5n+6$ odd

$n = 2K+1 \quad K \in \mathbb{Z}$

and $5(2K+1)+6 = 10K+5+6$

$= 10K + 11$

$= 2(\underbrace{5K+5}_{\text{some integer, call it } p}) + 1$

$= 2p+1$ and $5n+6$ is odd //

Now

$\longleftarrow$

pf/ $5n+6$ odd $\rightarrow$ $n \in \mathbb{Z}^+$ and odd

use contrapositive which says

$\neg(n \text{ odd}) \rightarrow \neg(5n+6 \text{ odd})$

i.e. n even $\rightarrow$ $5n+6$ even

Let $n = 2K$, $K \in \mathbb{Z}$

Now $5n+6 = 5(2K)+6$

$= 10K+6 = 2(\underbrace{5K+3}_{\text{some int. call it } p})$

$= 2p$

and $5n+6$ is even //

②

1.d) pf/

Let $r, s \in \mathbb{Q}$ so that $r = \frac{p_1}{q_1}$ and $s = \frac{p_2}{q_2}$

$p_1, p_2, q_1, q_2 \in \mathbb{Z}$ and $q_1, q_2 \neq 0$

$$r + s = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2}{q_1 q_2} + \frac{p_2 q_1}{q_1 q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

which is rational $\in \mathbb{Q}$

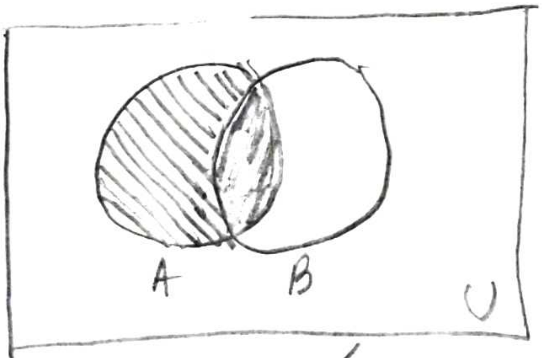
//

2. a) T
   b) T
   c) T
   d) F

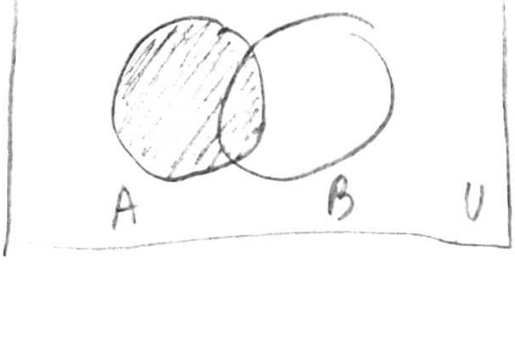
Note Typo! $\{\{0\}\} \subset \{\{0\},\{0\}\}$

3.
a) $A \cup B = \{0,1,2,3,4,5,6\}$
b) $A \cap B = \{3\}$
c) $B - A = \{0,6\}$

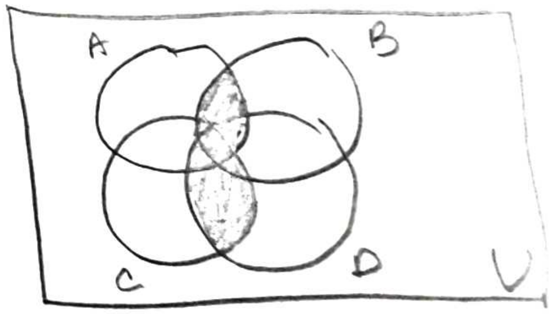
4. $(A \cap B) \cup (A \cap \bar{B}) = A$



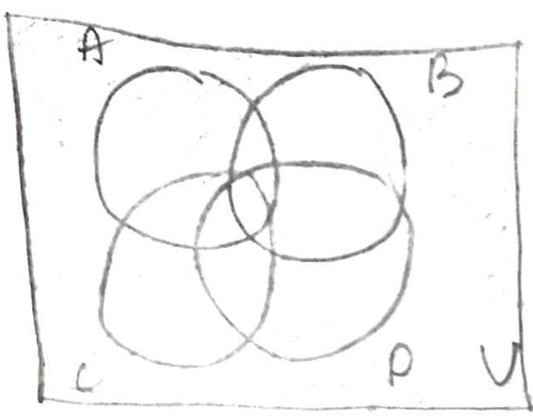
$\parallel$ ✓



5.
a)



b)



6. a) domain: $\mathbb{Z}^+ \times \mathbb{Z}^+$

range: $\mathbb{Z}^+$

b) domain: $\mathbb{Z}^+$

range: $\{0,1,2,3,4,5,6,7,8,9\}$

c) domain: set of bit strings

range: $\mathbb{N}$

d) domain: set of bit strings

range: $\mathbb{N}$

7. a) Yes

b) No

c) Yes

d) No

8. $f(x) = x^2+6 \quad g(x) = 2x+1$

$f \circ g(x) = f(g(x)) = f(2x+1) = (2x+1)^2+6 = 4x^2+4x+1+6$

$= \boxed{4x^2+4x+7}$

$g \circ f(x) = g(f(x)) = 2(x^2+6)+1$

$= 2x^2+12+1 = \boxed{2x^2+13}$

9. a) $\lfloor \frac{0}{2} \rfloor = 0$

$\lfloor \frac{1}{2} \rfloor = 0$

$\lfloor \frac{2}{2} \rfloor = 1$

$\lfloor \frac{3}{2} \rfloor = 1$

b) $\lfloor \frac{0}{2} \rfloor + \lceil \frac{0}{2} \rceil = 0$

$\lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil = 0+1 = 1$

$\lfloor \frac{2}{2} \rfloor + \lceil \frac{2}{2} \rceil = 1+1 = 2$

$\lfloor \frac{3}{2} \rfloor + \lceil \frac{3}{2} \rceil = 1+2 = 3$

10. a) $a_n = 1 + \frac{n(n+1)}{2}$

b) $a_n = n^2 + 4n + 4 = (n+2)^2$

c) $a_n = 1$

11. a) 10

b) 180

12. a) $a_n = 3 \cdot 2^{n-1}$; 384, 768, 1536; (or $a_n = 2 \cdot a_{n-1}$) $a_1 = 3$

b) $a_n = 15 - 7(n-1) = 22 - 7n$; -34, -41, -48; (or $a_n = a_{n-1} - 7$) $a_1 = 15$

c) $a_n = \frac{(n^2 + n + 4)}{2}$; 57, 68, 80; (or $a_n = a_{n-1} + n$) $a_1 = 3$

13 a) countably infinite

$\{0, -2, +2, -4, +4, \ldots\}$

$\begin{array}{ccccc} 0 & -2 & +2 & -4 & +4 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 \end{array}$

b) finite

c) uncountable

d) $\{(3,0), (4,0), (3,-1), (4,-1), (3,+1), (4,+1), \ldots\}$

$\begin{array}{cccccc} (3,0) & (4,0) & (3,-1) & (4,-1) & (3,+1) & (4,+1) \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$