Sample Exam #2

1. Prove the following statements:

   a. Every odd integer is the difference of two squares.

   b. If n is an integer and $n^3 + 5$ is odd, then n is even.

   c. If n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

   d. The sum of two rational numbers is rational.

2. Determine whether these statements are true or false.

   a. $\emptyset \in \{\emptyset\}$

   b. $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$

   c. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

   d. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

   a. $A \cup B$

   b. $A \cap B$

   c. $B - A$

4. Show that if A and B are sets, then

$$(A \cap B) \cup (A \cap \overline{B}) = A$$

5. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

   a. $(A \cap B) \cup (C \cap D)$

   b. $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$

6. Find the domain and range of these functions.

   a. the function that assigns to each pair of positive inte- gers the maximum of these two integers

   b. the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer

   c. the function that assigns to a bit string the number of times the block 11 appears

   d. the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s

7. Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

   a. $f(x) = 2x + 1$
   b. $f(x) = x^2 + 1$
   c. $f(x) = x^3$
   d. $f(x) = (x^2 + 1)/(x^2 + 2)$

8. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 6$ and $g(x) = 2x + 1$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

9. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

   a. $\lfloor n/2 \rfloor$?
   b. $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?

10. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach.

    a. $a_n = a_{n-1} + n$, $a_0 = 1$
    b. $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$
    c. $a_n = 2a_{n-1} - 1$, $a_0 = 1$

11. Find the value of each of these sums.

    a. $\sum_{j=0}^{8} (1 + (-1)^j)$
    b. $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

12. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

    a. $3, 6, 12, 24, 48, 96, 192, \ldots$
    b. $15, 8, 1, -6, -13, -20, -27, \ldots$

c. $3, 5, 8, 12, 17, 23, 30, 38, 47, \ldots$

13. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

    a. the even integers
    b. the integers less than 100
    c. the real numbers between 0 and 1
    d. the set $A \times \mathbb{Z}$ where $A = \{3, 4\}$