Sample Exam #1

1. What is the negation of this proposition? "If you pay your membership dues, then if you come to the club, you can enter free."

   Solution: The statement has the form $p \to (c \to e)$ where p stands for you pay your membership dues, c stands for you come to the club, and e stands for you can enter free. To more easily analyze the statement, we use De Morgans laws and other properties of predicates to rewrite it: $p \wedge \neg(c \to e)) \equiv p \wedge (c \wedge \neg e))$. In English the negation says that You pay your membership dues and you come to the club, but you cannot enter free.

2. Write the contrapositive, converse and inverse of the following proposition: "If the number is positive, then its square is positive."

   Solution: The contrapositive is the equivalent statement If the square of a number is not positive, then the number is not positive. The converse is If the square of a number is positive, then the number is positive. The inverse is If the number is not positive, then the square is not positive.

3. Suppose you have three cards: one red on both sides (red/red), one green on both sides (green/green), and one red on one side and green on the other side (red/green). The three cards are placed in a row on a table. Explain how to determine the identity of all three cards by selecting one card and turning it over.

   Solution: When the three cards are put in a row, exactly two of the three must have the same color showing say red. Pick one of these two red cards and turn it over. If the other side is also red, then you have found the red/red card. The other card with red showing must be the red/green card, and the card with green showing must be the green/green card. If the card you turn over has green on the other side, you have located the red/green card. The other card with red showing must be the red/red card, and the card with green showing must be the green/green card. (A similar procedure will determine the identity of the three cards, if two cards have green showing.)

4. Prove that $\neg[r \vee (q \wedge (\neg r \to \neg p))] \equiv \neg r \wedge (p \vee \neg q)$ by using a truth table.

5. Suppose you want to prove a theorem of the form $p \to (q \vee r)$. Prove that this is equivalent to showing that $(p \wedge \neg q) \to r$.

   Solution: We will give a proof by replacing the first statement by equivalent statements until finally the second statement is obtained. (We will use the equivalence $a \to b \equiv \neg a \vee b$ twice.)

$$\begin{aligned} p \to (q \vee r) \quad &\equiv \neg p \vee (q \vee r) \\ &\equiv (\neg p \vee q) \vee r \\ &\equiv \neg(p \wedge \neg q) \vee r \\ &\equiv (p \wedge \neg q) \to r. \end{aligned}$$

   We could also give a proof by constructing a truth table.

6. Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

   Solution:

$$\begin{aligned}(p \to q) \vee (p \to r) &\equiv (\neg p \vee q) \vee (\neg p \vee r)\\ &\equiv (\neg p \vee \neg p) \vee (q \vee r)\\ &\equiv \neg p \vee (q \vee r)\\ &\equiv p \to (q \vee r).\end{aligned}$$

   We could also give a proof by constructing a truth table.

7. Let $Q(x, y)$ be the statement $x + y = x - y$ where the universe for x and y is the set of all real numbers. Determine the truth value of:

   (a) $Q(5, -2)$

   (b) $Q(4.7, 0)$

   (c) Determine the set of all pairs of numbers, x and y, such that $Q(x, y)$ is true.

   Solution:

   (a) $Q(5, -2)$ says that $5 + (-2) = 5 - (-2)$, or $3 = 7$, which is false.

   (b) $Q(4.7, 0)$ says that $4.7 + 0 = 4.7 - 0$, which is true.

   (c) $x + y = x - y$ if and only if $x + 2y = x$, which is true if and only if $y = 0$. Therefore, x can be any real number and y must be zero.

8. Write the following statement in English, using the predicates
   $C(x)$: "x is a Computer Science major"
   $M(y)$: "y is a math course"
   $T(x, y)$: "x is taking y"
   where x represents students and y represents courses:

$$\forall x \exists y (C(x) \to M(y) \wedge T(x, y)).$$

   Solution: The statement $\forall x \exists y (C(x) \to M(y) \wedge T(x, y))$ asserts that for every student x there is a course y such that if x is a major in Computer Science then x is taking y and y is a math course. Therefore, Every Computer Science major is taking at least one math course.

9. Determine whether this argument is valid:
   It is not sunny this afternoon and it is colder than yesterday.
   If we go swimming, then it is sunny.
   If we do not go swimming, then we will take a canoe trip.

If we take a canoe trip, then we will be home by sunset.
Therefore we will be home by sunset.

Solution:

(a) $\neg p \wedge q$ Premise
(b) $r \to p$ Premise
(c) $\neg r \to s$ Premise
(d) $s \to t$ Premise
(e) $\neg p$ Simplification using 1.
(f) $\neg r$ Modus tollens using 2. and 5.
(g) s Modus ponens using 3. and 6.
(h) t Modus ponens using 4. and 7.

10. Suppose we have the two propositions (with symbols to represent them):
It is raining (r) or I work in the yard (w)
It is not raining ($\neg r$) or I go to the library (l).
What conclusion can we draw from these propositions?

Solution: We can use the resolution rule of inference to draw a conclusion from these propositions. In symbols the two given propositions are $(r \vee w) \wedge (\neg r \vee l)$. We can use the resolution rule of inference to draw a conclusion from these propositions. From resolution we have $(r \vee w) \wedge (\neg r \vee l) \to (w \vee l)$. Therefore we can draw the conclusion, $(w \vee l)$, I work in the yard or I go to the library.

11. Determine the truth value of each of these statements if the doman consists of **all integers**.

(a) $\forall n(n^2 \geq 0)$
(b) $\exists n(n^2 = 2)$
(c) $\forall n(n^2 \geq n)$
(d) $\exists n(n^2 < 0)$

Solution:

(a) T
(b) F
(c) T
(d) F

12. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

(a) $\forall x(x^2 \geq x)$

(b) $\forall x(x > 0 \lor x < 0)$

(c) $\forall x(x = 1)$

Solution:

(a) $x = .25$.

(b) $x = 0$

(c) $x = 2$

13. Let $L(x, y)$ be the statement x loves y, where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

(a) Everybody loves Jerry.

(b) Everybody loves somebody.

(c) There is somebody whom everybody loves.

(d) Nobody loves everybody.

(e) There is somebody whom Lydia does not love.

Solution:

(a) $\forall x L(x, Jerry)$

(b) $\forall x \exists y L(x, y)$

(c) $\exists y \forall x L(x, y)$

(d) $\forall x \exists y \neg L(x, y)$

(e) $\exists x \neg L(Lydia, x)$