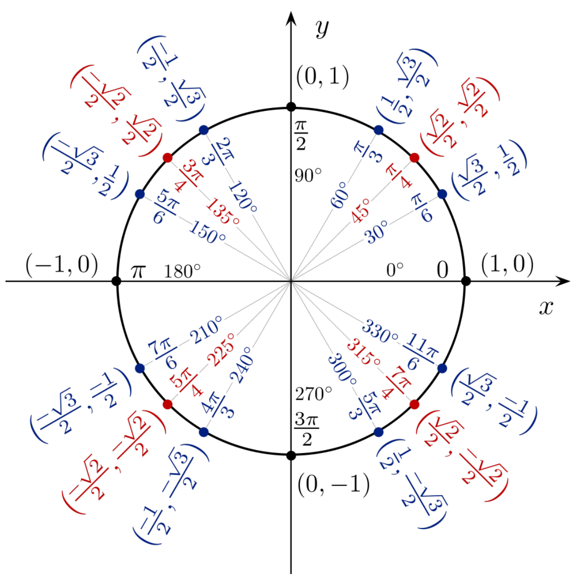
Mathematics 1375/D572, Spring 2019 **Exam #4 (Take-home)**
Instructor: Suman Ganguli **Due: Friday, May 17** **Name:** ______________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	12	5	10	5	5	5	5	50
Score:									

In order to receive full credit, you must **show all your work** and simplify your answers. You must **work on this exam alone**, but you should study resources such as your class notes, WebWork exercises, the Final Exam Review exercises, the textbook, etc.

1. (3 points) Recall the following graph of the unit circle, which shows the coordinates of points on the unit circle at certain "special angles":



(a) We can use this graph of the unit circle to find solutions of *some* trigonometric equations of the form

$\sin\theta = c$ or $\cos\theta = c$

For which values of c will this graph work?

Solution:

$$c = 0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1$$

(b) Describe, step by step, the procedure by which we can use this graph of the unit circle to find *all* solutions of such trigonometric equations $\sin\theta = c$ or $\cos\theta = c$:

Solution:

- For $\sin\theta = c$, look up all angles θ on the unit circle where the y-coordinate is c. For $\cos\theta = c$, look up all angles θ on the unit circle where the x-coordinate is c.
- The list of all solutions is $\theta \pm 2\pi n$ for $n = 0, \pm1, \pm2, \ldots$ (for each θ from the 1st step)

2. (12 points) Find all solutions (in radians) of the following trigonometric equations. For each equation:

- show all preliminary algebraic steps
- explain how you use the unit circle to find the solutions in $[0, 2\pi)$
- and then write down expressions giving all solutions

(a) $8\sin\theta + 4 = 0$

Solution: $8\sin\theta + 4 = 0 \Longleftrightarrow \sin\theta = -\frac{1}{2}$, so we look for the angles θ for which the y-coordinate of the point on the unit circle is $-\frac{1}{2}$. These are $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$, and so the solutions of the equation are

$$\theta = \frac{7\pi}{6} + 2\pi n$$

$$\theta = \frac{11\pi}{6} + 2\pi n$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(b) $\tan^2\theta - 1 = 0$ (Hint: use $u^2 - 1 = (u-1)(u+1)$.)

Solution: By factoring $\tan^2\theta - 1 = (\tan\theta - 1)(\tan\theta + 1)$ We need to find the angles θ such that $\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm 1$, i.e, where $\sin\theta = \cos\theta$ or $\sin\theta = -\cos\theta$. So we look for the points on the unit circle where the x- and y-coordinates are equal or equal but of opposite sign.
$\sin\theta = \cos\theta$ at $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$, and $\sin\theta = -\cos\theta$ at $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$, and so the complete solution set is

$$\theta = \frac{\pi}{4} + 2\pi n$$

$$\theta = \frac{3\pi}{4} + 2\pi n$$

$$\theta = \frac{5\pi}{4} + 2\pi n$$

$$\theta = \frac{7\pi}{4} + 2\pi n$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(c) $2\cos^2\theta - \cos\theta = 0$ (Hint: use $2u^2 - u = u(2u-1)$.)

Solution: Since $2\cos^2\theta - \cos\theta = \cos\theta(2\cos\theta - 1)$, the solution set of the given equation consists of the solutions of $\cos\theta = 0$ and the solutions of $2\cos\theta - 1$, i.e., $\cos\theta = \frac{1}{2}$.
The solutions of $\cos\theta = 0$ are $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.
From the unit circle, we see that $\cos\theta = 0$ for $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, and $\cos\theta = \frac{1}{2}$ for $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.
Thus, the complete solution set of $2\cos^2\theta - \cos\theta = 0$ is

$$\theta = \frac{\pi}{2} \pm 2\pi n \text{ for } n = 0, 1, 2, 3, \ldots$$

$$\theta = \frac{3\pi}{2} \pm 2\pi n \text{ for } n = 0, 1, 2, 3, \ldots$$

$$\theta = \frac{\pi}{3} \pm 2\pi n \text{ for } n = 0, 1, 2, 3, \ldots$$

$$\theta = \frac{5\pi}{3} \pm 2\pi n \text{ for } n = 0, 1, 2, 3, \ldots$$

3. (5 points) The mathematical model for exponential growth or decay of some quantity Q is given by the function

$$Q(t) = Q_0(1 + r)^t$$

where the independent variable t typically represents time.

(a) What do the parameters (i.e., constants) in the function $Q(t)$ represent?

Solution:

- Q_0 represents the initial quantity, i.e., the quantity at time $t = 0$
- r represents the growth rate, i.e., the percentage amount the quantity grows over each time period

(b) What determines whether Q is undergoing exponential growth or exponential decay? (Hint: your explanation should use simple inequalities involving one of the parameters.)

Solution: Q is undergoing exponential growth if $r > 0$, and undergoing exponential decay if $r < 0$. Note that if $r = 0$, then $Q(t) = Q_0$, i.e., Q neither grows nor decays, but just remains constant at Q_0.

4. (10 points) The world's population is approximately 7 billion people at present, and is estimated to be growing at a rate of 1.2% per year.

(a) Assuming the growth rate stays constant, write down the formula for $P(t)$ = the world population t years from now:

Solution: Since $r = 1.2\% = 0.012, 1 + r = 1.012$ and so $P(t) = 7(1.012)^t$ (where for simplicity we are using units of billions of people for $P(t)$)

(b) Using these assumptions, what will the world population be in 2050 (i.e., 31 years from now)?

Solution: $P(31) = 7(1.012)^{31} = 10.1319707564$
So under these assumptions, the world population will be $10,131,970,756$ in 2050.

(c) Under these assumptions, in what year will the world population be double what it is now? Show all algebra.

Solution: Since the current populations is given as 7 billion, we need to find the year in which the population will reach 14 billion, i.e., we need to solve for t such that:

$$P(t) = 7(1.012)^t = 14$$

Dividing through by 7 and taking the logarithm of both sides yields:

$$t \log(1.012) = \log 2 \Longrightarrow t = \frac{\log(2)}{\log 1.012} =\approx 58.1$$

So according to these assumptions, the world population will double to 14 billion in approximately 58 years, i.e., in the year $2019 + 58 = 2077$.
Note: these assumptions are in fact not realistic. Although the world population's growth rate is currently close to 1.12%, it is not constant. Via `http://www.worldometers.info/world-population/#growthrate`:

> Population in the world is currently (as of 2015-2016) growing at a rate of around 1.13% per year. The average population change is currently estimated at around 80 million per year.
>
> Annual growth rate reached its peak in the late 1960s, when it was at 2% and above. The rate of increase has therefore almost halved since its peak of 2.19 percent, which was reached in 1963.
>
> The annual growth rate is currently declining and is projected to continue to decline in the coming years. Currently, it is estimated that it will become less than 1% by 2020 and less than 0.5% by 2050.

5. (5 points) Consider a trigonometric function of the form $f(x) = a\sin(bx+c)$ or $f(x) = a\cos(bx+c)$.

Give the formula for calculating each of the following quantities (in terms of the parameters a, b, c) and briefly describe what each quantity represents on the graph of the trig function:

- amplitude of $f =$

Solution: The amplitude of f is $|a|$, and represents how high each peak of the graph is on the y-axis.

- period of $f =$

Solution: The period of f is $\left|\dfrac{2\pi}{b}\right|$, and represents the length of the interval on the x-axis that it takes for the f to complete one full cycle of the trig function.

- phase shift:

Solution: The period of f is $-\dfrac{c}{b}$, and represents the point on the x-axis where the trig function begins it's first full cycle.

6. (5 points) Consider the function $f(x) = -5\sin(2x - \pi)$.

(a) Find the amplitude, period and phase shift of $f(x)$:

- amplitude:

Solution: $|a| = |-5| = 5$

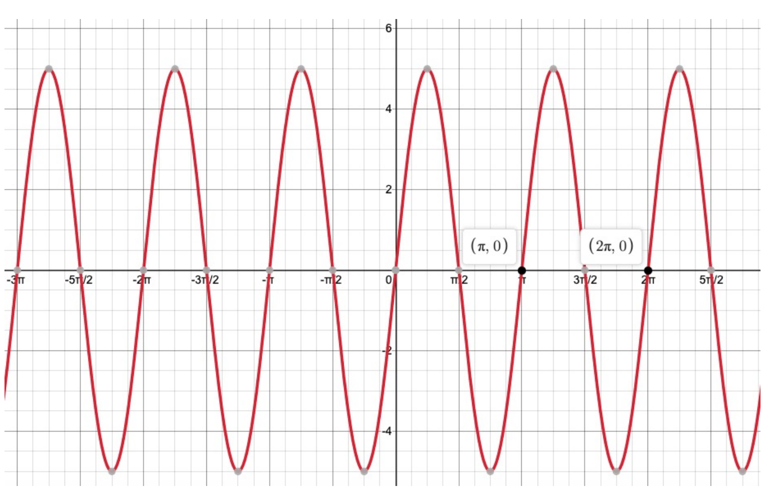
- period:

Solution: $\left|\dfrac{2\pi}{b}\right| = \dfrac{2\pi}{2} = \pi$

- phase shift:

Solution: $-\dfrac{c}{b} = -\dfrac{-\pi}{2} = \dfrac{\pi}{2}$

(b) Sketch the graph of $f(x)$. Label the amplitude, period and phase shift on the graph.



7. (5 points) An arithmetic sequence is a sequence of numbers $a_1, a_2, a_3, a_4, \ldots$ where there is a constant "difference" d between successive terms, i.e.,

$$a_2 = a_1 + d, a_3 = a_2 + d, \ldots$$

The general formula for the n-th term in an arithmetic sequence is:

$$a_n = a_1 + d(n-1)$$

and the formula for the sum of the first k terms in an arithmetic sequence is:

$$a_1 + a_2 + a_3 + \ldots + a_k = \frac{k(a_1 + a_k)}{2}$$

Now consider the arithmetic sequence

$$a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, \ldots$$

(a) Apply the general formula to calculate the 40th term in this arithmetic sequence (i.e., you should **not** calculate all the terms $a_5, a_6 \ldots a_{39}$!)

Solution:

$$a_{40} = 4 + 3(39) = 4 + 117 = 121$$

(b) Apply the formula to find the sum of the first 40 terms in this arithmetic sequence:

Solution:

$$a_1 + a_2 + a_3 + \ldots + a_{40} = \frac{40(4 + 121)}{2} = 2500$$

8. (5 points) A geometric sequence is a sequence of numbers $a_1, a_2, a_3, a_4, \ldots$ where there is a constant "ratio" r between successive terms, i.e.,

$$a_2 = a_1 * r, a_3 = a_2 * r, \ldots$$

For a geometric series with $-1 < r < 1$, the infinite geometric sum is given by the formula

$$a_1 + a_2 + a_3 + \ldots = \frac{a_1}{1-r}$$

Now consider the geometric sequence

$$a_1 = 5, a_2 = -\frac{5}{3}, a_3 = \frac{5}{9}, a_4 = -\frac{5}{27}, \ldots$$

(a) For this geometric sequence, the ratio r is

Solution:

$$r = -\frac{1}{3}$$

(b) Calculate the sum of this infinite geometric sequence:

Solution:

$$a_1 + a_2 + a_3 + \ldots = \frac{5}{1-(-1/3)} = \frac{5}{(4/3)} = \frac{15}{4}$$