Worksheet 5.1 - Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION: To prove that $P(n)$ is true for all positive integers n, where$P(n)$ is a propositional function, we complete two steps:

BASE STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**.

1. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for the positive integer n.
   (a) What is the statement $P(1)$?
   (b) Show that $P(1)$ is true, completing the base case of the proof.
   (c) What is the inductive hypothesis?
   (d) What do you need to prove in the inductive step?
   (e) Complete the inductive step, identifying where you use the inductive hypothesis.
   (f) Explain why these steps show that this formula is true whenever n is a positive integer.

2. The following puzzle is an exerpt from

   http://pi.math.cornell.edu/ mec/2008-2009/ABjorndahl/ppmi.pdf

   After a particularly successful heist, a group of criminal ex-mathematicians found themselves in possession of 1 million dollars. A meeting was called to distribute the money to the members. Exactly how many members there are is not known outside of the group itself, so we'll just have to say that there are n members and leave it at that. What is known is that each member has a unique rank in the organization, from 1st ranked (the leader) all the way down to nth ranked (the last-in-command).

   As it turns out, a very precise Code is in place that governs how surplus income is to be distributed. To begin with, the 1st ranked member decides on a potential distribution of the wealth. Each member must be assigned a whole dollar amount (no cents), with 0 dollars of course being allowed. This potential distribution is then put to a secret vote, wherein each member, including the leader, gets to cast exactly one ballot: "Yes" or "No." The members cannot communicate or strategize amongst themselves; it is every ex-mathematician for themselves.

   If the vote passes or is a tie, then the money is distributed according to the proposed distribution. The catch is this: if the vote fails, then the 1st ranked member is ousted from the organization forever. Every other member is promoted by exactly one rank to fill the power vacuum, and the new 1st ranked member (who used to be 2nd ranked) repeats the process by indicating a new potential distribution and putting it to a vote. This continues until one of the distributions is passed, at which point the members take whatever money was allotted to them by that distribution.

Each member is very invested in this international network, and would rather get no share of the money at all than be ousted from the organization. Each member would also prefer not to oust too many people, if possible, so if all else is equal (i.e. if they would get the same payoff either way), then a member will vote "Yes" rather than "No" on a given distribution. Of course, if they figure that they can get even a single extra dollar by voting "No" on the current plan, they will do it. That's the way the world works, at least among secret math thieves.

(a) Let n represent the number of members of the network. If $n = 1$, how much cash can the leader pocket? Denote this by $P(1)$.

(b) Let $n = 2$ and $n = 3$, how much cash can the leader pocket then? In other words, what is $P(2)$ and $P(3)$)?

(c) In general, for any n: how much cash can the leader pocket, $P(n)$?

(d) Explain, in words, how a proof by induction of your statement $P(n)$ works:

i. What is the base case?

ii. What is your inductive hypothesis?

iii. What do you need to prove in the inductive step?

iv. Complete the inductive step, identifying where you use the inductive hypothesis.

v. Explain why these steps show $P(n)$ is true whenever n is a positive integer.

3. (a) Find a formula for

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

by examining the values of this expression for small values of n.

(b) Prove the formula you conjectured in part (a).