Handout 4.1 & 4.2

DEFINITION 1 If a and b are integers with $a \neq 0$, we say that a *divides* b if there is an integer c such that $b = ac$, or equivalently, if $\frac{b}{a}$ is an integer. When a divides b we say that a is a *factor* or *divisor* of b, and that b is a *multiple* of a. The notation $a \mid b$ denotes that a divides b. We write $a \nmid b$ when a does not divide b.

THEOREM 1 Let a, b, and c be integers, where $a \neq 0$. Then

(*i*) if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;
(*ii*) if $a \mid b$, then $a \mid bc$ for all integers c;
(*iii*) if $a \mid b$ and $b \mid c$, then $a \mid c$.

COROLLARY 1 If a, b, and c are integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ whenever m and n are integers.

DEFINITION 2 In the equality given in the division algorithm, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*. This notation is used to express the quotient and remainder:

$$q = a \textbf{ div } d, \quad r = a \textbf{ mod } d.$$

THEOREM 2 **THE DIVISION ALGORITHM** Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$.

DEFINITION 3 If a and b are integers and m is a positive integer, then a is *congruent to b modulo m* if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m. We say that $a \equiv b \pmod{m}$ is a **congruence** and that m is its **modulus** (plural **moduli**). If a and b are not congruent modulo m, we write $a \not\equiv b \pmod{m}$.

THEOREM 3 Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \textbf{ mod } m = b \textbf{ mod } m$.

THEOREM 4 Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

THEOREM 5 Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m} \qquad \text{and} \qquad ac \equiv bd \pmod{m}.$$

COROLLARY 2 Let m be a positive integer and let a and b be integers. Then

$$(a + b) \textbf{ mod } m = ((a \textbf{ mod } m) + (b \textbf{ mod } m)) \textbf{ mod } m$$

and

$$ab \textbf{ mod } m = ((a \textbf{ mod } m)(b \textbf{ mod } m)) \textbf{ mod } m.$$

THEOREM 1 Let b be an integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0,$$

where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$.

ALGORITHM 1 Constructing Base b Expansions.

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while $q \neq 0$
  $a_k := q$ **mod** b
  $q := q$ **div** b
  $k := k + 1$
return $(a_{k-1}, \ldots, a_1, a_0)$ $\{(a_{k-1} \ldots a_1 a_0)_b$ is the base b expansion of $n\}$