MAT2440/D641 **Exam #4 (Take-Home)**
Instructor: Suman Ganguli **Due: Monday, December 17** **Name:** ____________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Consider the function $f(n)$ which takes as input any integer n and returns n **mod** 6, i.e.,

$$f(n) = n \textbf{ mod } 6$$

(a) Calculate the following:

i. $f(13) =$

Solution: $f(13) = 13 \textbf{ mod } 6 = 1$ (since $13 = 6(2) + 1$)

ii. $f(54) =$

Solution: $f(54) = 54 \textbf{ mod } 6 = 0$ (since $54 = 6(9) + 0$)

iii. $f(-7) =$

Solution: $f(-7) = -7 \textbf{ mod } 6 = 5$ (since -7 = 6(-2) + 5)

(b) What is the domain of the function f? What is its range?

Solution: The domain is $\mathbb{Z}$ (the set of all integers) and the range is $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

(c) Show that f is *not* one-to-one.

Solution:
f is clearly not one-to-one, since there are many pairs of integers which have the same remainder when divided by 6. Just take any two integers which are congruent modulo 6, i.e., whose difference is divisible by 6 (see the definition on p240 of the textbook).
For example, 13 and 19 are congruent modulo 6, and:

$$f(13) = 13 \textbf{ mod } 6 = 1$$
$$f(19) = 19 \textbf{ mod } 6 = 1$$

2. (10 points) For this question, use predicate logic in the language of arithmetic, i.e., using the quantifiers ($\exists$ and $\forall$), the standard logical connectives ($\wedge, \vee, \neg, \rightarrow$), and the standard symbols/predicates of arithmetic ($=, +, \cdot, <, >$). Assume that the domain of discourse for the quantifiers is the set of integers $\mathbb{Z}$.

(a) The following is a restatement of the Division Algorithm (from p239 of the textbook):

The Division Algorithm: "For any integers a and any positive integer d, there are integers q and r, with $0 \leqslant r < d$, such that $a = d \cdot q + r$."

Express the Division Algorithm as a statement of predicate logic in the language of arithmetic:

Solution:

$$\forall a \forall d((d > 0) \rightarrow \exists q \exists r((r = 0) \vee (r > 0) \wedge (r < d) \wedge (a = d \cdot q + r))$$

(b) Translate the following statement of predicate logic into an English sentence about a given integer p:

$$\forall n((0 < n \wedge n \mid p) \rightarrow (n = 1 \vee n = p))$$

(Recall that "$n \mid p$" is the notation for "n is a factor of p"; see p238 of the textbook. You can use that phrase in your answer.)

Solution:
"Every positive integer n which is a factor of p is either equal to p or equal to 1."
Or more succinctly: "The only positive factors of p are 1 and p."
Note that this is the definition of p being a prime number, which is a fundamental concept in number theory. See pp257-262 in Sec 4.3 of the textbook.

3. (10 points) Here is pseudocode which implements the Division Algorithm (as shown and explained on p253 of the textbook):

procedure `division-algorithm` (a : integer, d: positive integer)
  $q := 0$
  $r := |a|$
  while ($r \geqslant d$):
    $r := r - d$
    $q := q + 1$
  if ($a < 0$ and $r > 0$) **then**
    $r := d - r$
    $q := -(q + 1)$
  return (q, r)

List the steps used by this implementation of the Division Algorithm to compute -19 **mod** 5 and -19 **div** 5:

Solution:

- **Step 0:** The inputs to the algorithm are $a = -19$ and $d = 5$.

  The algorithm initially sets $q = 0$ and $r := |a| = |-19| = 19$.

- **Step 1:** Since $r = 19 \geqslant d = 5$, the algorithm enters the **while** loop. The updated values of r and q are

$$r := r - d = 19 - 5 = 14 \text{ and } q := q + 1 = 0 + 1 = 1$$

- **Step 2:** Since $r = 14 \geqslant d = 5$, the algorithm again enters the **while** loop. The updated values of r and q are

$$r := r - d = 14 - 5 = 9 \text{ and } q := q + 1 = 1 + 1 = 2$$

- **Step 3:** Since $r = 9 \geqslant d = 5$, the algorithm again enters the **while** loop. The updated values of r and q are

$$r := r - d = 9 - 5 = 4 \text{ and } q := q + 1 = 2 + 1 = 3$$

- **Step 4:** Since $r = 4 < d = 5$, the algorithm does *not* enter the **while** loop again. Since $a = -19 < 0$ and $r = 4 > 0$, then algorithm executes the **if-then** statements, and updates r and q to

$$r := d - r = 5 - 4 = 1 \text{ and } q := -(q + 1) = -(3 + 1) = -4$$

  So the algorithm returns $(q, r) = (-4.1)$, and indeed -19 **mod** $5 = 1$ and -19 **div** $5 = -4$ (since $-19 = 5(-4) + 1$).

4. (10 points) Use mathematical induction to prove that for all positive integers n, i.e., for all $n \geqslant 1$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

according to the two steps outlined below.

(Hint: Read Sec. 5.1, pp312-317 of the textbook, especially Examples 1 & 2 on pp316-317.)

(a) Let $P(n)$ be the proposition that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$.

The **basis step (or base case)** is to show that $P(1)$ is true. Write down $P(1)$ and show that it is true.

Solution: $P(1)$ is the equation

$$\frac{1}{1\cdot 2}=\frac{1}{1+1}$$

which is obviously true, since both sides of the equation equal $\frac{1}{2}$.

(b) For the **inductive step**, assume that $P(k)$ is true, i.e., assume

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$$

and show that this implies that $P(k+1)$ is true.

Solution: Assume that $P(k)$ is true:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

We need to show $P(k+1)$ is true:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Starting with the LHS, and using the assumption that $P(k)$ is true:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}\cdot\frac{k+2}{k+2}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)(k+1)}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

5. (10 points) Consider a compound proposition ϕ of propositional logic. (See Sec 1.1 of the textbook to review the definition of a compound proposition.)

   (a) Write down, **in terms of the truth table for $\boldsymbol{\phi}$**, definitions of what it means for ϕ to be:

   i. a tautology:

   Solution: A compound proposition ϕ is a tautology if *every* entry in its truth table (i.e., every entry in the column under ϕ) is T.

   ii. satisfiable:

   Solution: A compound proposition ϕ is satisfiable if *some* entry in its truth table is T.

   iii. unsatisfiable:

   Solution: A compound proposition ϕ is unsatisfiable if every entry in its truth table is F.

   (b) Prove that if ϕ is unsatisfiable, then $\neg\phi$ is a tautology.

   Solution: Assume ϕ is an unsatisfiable compound proposition. Then, by definition, every entry in its truth table is F. But then every entry in the truth table for $\neg\phi$ is T, since negation "flips" each F entry to T. Thus, by the definition, $\neg\phi$ is a tautology.