MAT2440/D641 **Exam #4 (Take-Home)**
Instructor: Suman Ganguli **Due: Monday, December 17** **Name:** ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Consider the function $f(n)$ which takes as input any integer n and returns n **mod** 6, i.e.,

$$f(n) = n \textbf{ mod } 6$$

(a) Calculate the following:

i. $f(13) =$

ii. $f(54) =$

iii. $f(-7) =$

(b) What is the domain of the function f? What is its range?

(c) Show that f is *not* one-to-one.

2. (10 points) For this question, use predicate logic in the language of arithmetic, i.e., using the quantifiers ($\exists$ and $\forall$), the standard logical connectives ($\wedge, \vee, \neg, \rightarrow$), and the standard symbols/predicates of arithmetic ($=, +, \cdot, <, >$). Assume that the domain of discourse for the quantifiers is the set of integers $\mathbb{Z}$.

   (a) The following is a restatement of the Division Algorithm (from p239 of the textbook):

   The Division Algorithm: "For any integers a and any positive integer d, there are integers q and r, with $0 \leqslant r < d$, such that $a = d \cdot q + r$."

   Express the Division Algorithm as a statement of predicate logic in the language of arithmetic:

   (b) Translate the following statement of predicate logic into an English sentence about a given integer p:

   $$\forall n((0 < n \wedge n \mid p) \rightarrow (n = 1 \vee n = p))$$

   (Recall that "$n \mid p$" is the notation for "n is a factor of p"; see p238 of the textbook. You can use that phrase in your answer.)

3. (10 points) Here is pseudocode which implements the Division Algorithm (as shown and explained on p253 of the textbook):

procedure `division-algorithm` (a : integer, d: positive integer)

  $q := 0$

  $r := |a|$

  while ($r \geqslant d$):

    $r := r - d$

    $q := q + 1$

  if ($a < 0$ and $r > 0$) **then**

    $r := d - r$

    $q := -(q + 1)$

  return (q, r)

List the steps used by this implementation of the Division Algorithm to compute -19 **mod** 5 and -19 **div** 5:

- **Step 0:** In order to compute -19 **mod** 5 and -19 **div** 5, the inputs to the algorithm are

  $a =$ ______________ and $d =$ ______________

  The algorithm initially sets $q = 0$ and $r := |a| =$ ______________.

- **Step 1:** Since $r \geqslant d$, the algorithm enters the **while** loop. The updated values of r and q after the first loop are:

  $$r := r - d = \underline{\hspace{3cm}} \text{ and } q := q + 1 = \underline{\hspace{3cm}}$$

- Show the steps of the algorithm for each subsequent time it enters the **while** loop, i.e., write down the calculations and updated values of r and q after each loop:

- Why does the algorithm escape (i.e., not enter) the **while** loop? What does the algorithm do then? What values does the algorithm return?

4. (10 points) Use mathematical induction to prove that for all positive integers n, i.e., for all $n \geqslant 1$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

according to the two steps outlined below.

(Hint: Read Sec. 5.1, pp312-317 of the textbook, especially Examples 1 & 2 on pp316-317.)

(a) Let $P(n)$ be the proposition that $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots+\dfrac{1}{n\cdot(n+1)}=\dfrac{n}{n+1}$.

The **basis step (or base case)** is to show that $P(1)$ is true. Write down $P(1)$ and show that it is true.

(b) For the **inductive step**, assume that $P(k)$ is true, i.e., assume

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$$

and show that this implies that $P(k+1)$ is true.

5. (10 points) Consider a compound proposition ϕ of propositional logic. (See Sec 1.1 of the textbook to review the definition of a compound proposition.)

   (a) Write down, **in terms of the truth table for ϕ**, definitions of what it means for ϕ to be:

   - a tautology:

   - satisfiable:

   - unsatisfiable:

   (b) Prove that if ϕ is unsatisfiable, then $\neg\phi$ is a tautology.