**MAT2440/D641** **Exam #2**
**Instructor: Suman Ganguli** **Wednesday, October 31** **Name:** ____________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 10 | |
| Total: | 50 | |

1. (10 points) Recall the definitions of rational and irrational real numbers:

**Definition:** A real number $x$ is *rational* if there exist integers $a$ and $b$ such that $x = \frac{a}{b}$. A real number is *irrational* if it not rational.

Use the definition to prove the following theorems:

a. **Theorem:** If $x$ is rational and $x \neq 0$, then $1/x$ is also rational.
(Hint: Provide a direct proof.)
**Proof:**

**Solution:** For a direct proof, we begin by assume the hypothesis is true. So a proof is as follows:
Assume that $x$ is a non-zero rational real number. By definition (of being rational), $x = \frac{a}{b}$ for integers $a$ and $b$. But then

$$\frac{1}{x} = \frac{1}{a/b} = \frac{b}{a}$$

So $1/x$ is also a ratio of two integers, and thus is also a rational number.

b. **Theorem:** If $x$ is irrational, then $1/x$ is also irrational.
(Hint: Provide a indirect proof, i.e., a proof by contraposition or by contradiction.)
**Proof:**

**Solution:** For a proof by contraposition, we begin by assuming the negation of the conclusion. So:
Assume that $1/x$ is *not* irrational, i.e., that $1/x$ is rational. That means

$$\frac{1}{x} = \frac{a}{b}$$

for integers $a$ and $b$. But if we solve this equation for $x$, we see that

$$x = \frac{b}{a}$$

which means $x$ is rational. This proves the theorem (since we have shown the negation of the conclusion implies the negation of the hypothesis).

2. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{0, 1\}$.

   a. (10 points) List the elements of the following sets:

      i. $A \cup B =$

      **Solution:** $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$

      ii. $A \cap B =$

      **Solution:** $A \cap B = \{1\}$

      iii. $A - B =$

      **Solution:** $A - B = \{2, 3, 4, 5, 6\}$

      iv. $B - A =$

      **Solution:** $B - A = \{0\}$

      v. $A \times B =$

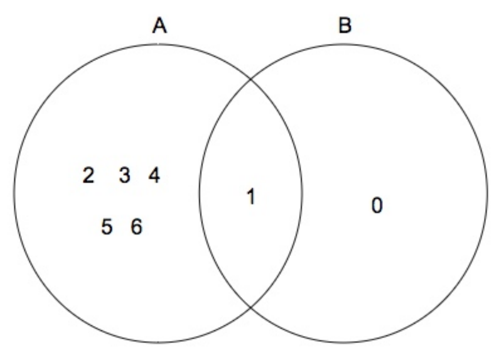
      **Solution:** $A \times B = \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1), (5, 0), (5, 1), (6, 0), (6, 1)\}$

      vi. (Extra credit!) $\mathcal{P}(A) \cap \mathcal{P}(B) =$

      **Solution:** $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing, \{1\}\}$

   b. (5 points) Draw a Venn diagram illustrating the sets $A$ and $B$, representing all of their elements with points in the appropriate regions in the diagram.

   **Solution:**




   c. (5 points) Consider the function $f : A \times A \to \mathbb{N}$ defined by the formula $f(a_1, a_2) = a_1 + a_2$.

      i. What is the range of $f$?

      **Solution:** Note that the maximum value of $f$ is $f(6, 6) = 6 + 6 = 12$, and the minimum value is $f(1, 1) = 1 + 1 = 2$. All the integers between 2 and 12 are also in the range, since each such integer is the sum of a pair of numbers from $A$. Thus, the range of $f$ is the set $\{$ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 $\}$.

      ii. Show that $f$ is *not* a one-to-one function.

      **Solution:** In order to show $f$ is not one-to-one, we need to find two distinct inputs in the domain $A \times A$ which get mapped by $f$ to the same output in the range. There are many such examples for this function; for instance $f(3, 1) = 3 + 1 = 4$ and $f(2, 2) = 2 + 2 = 4$.

3. Consider the following definition:

**Definition:** If $a$ and $b$ are integers, we say that $a$ *divides* $b$ if there is an integer $j$ such that $b = a * j$, or equivalently, if $\frac{b}{a}$ is an integer $j$. We also say $a$ is a *factor* of $b$, and $b$ is a *multiple* of $a$.

**Examples:** 3 divides 12 since $12/3 = 4$ is an integer, i.e., $12 = 3 * j$ for $j = 4$. But 5 does not divide 12, since $12/5$ is not an integer.

a. (5 points) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. List the elements of the following subsets of $U$:

i. $A = \{n \in U \mid 3 \text{ divides } n\} =$

**Solution:** $A$ consists of the integer multiples of 3 in $U$, i.e., $A = \{3, 6, 9, 12\}$

ii. $B = \{n \in U \mid n \text{ divides } 12\} =$

**Solution:** $B$ consists of all the factors of 12 in $U$, i.e., $B = \{1, 2, 3, 4, 6, 12\}$

b. (5 points) Provide a proof of the following theorem:
**Theorem:** If $a$ divides $b$ and $a$ divides $c$, then $a$ also divides $b + c$.
(Hint: Provide a direct proof, and *use the definition* given above.)
**Proof:**

**Solution:** We use the notation $a \mid b$ for "$a$ divides $b$" (this is standard notation that we will introduce in later in the semester, when we study number theory; see Section 4.1 of the textbook.)
For a direct proof of the theorem, assume $a \mid b$ and $a \mid c$. Then, by the definition, $b = a * j$ and $c = a * k$ for some integers $j$ and $k$. Then $b + c = a * j + a * k = a * (j + k)$, which establishes that $a \mid (b + c)$.

4. (10 points) Let $A = \{a, b, c\}$.

a. List the elements of the power set of $A$. (Hint: Since $A$ has 3 elements, its power set has $2^3 = 8$ elements.)

$\mathcal{P}(A) =$

**Solution:** $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

b. Construct a function $f : A \to \mathcal{P}(A)$ which has the following properties:

1. $f$ is one-to-one, and
2. $\forall x \in A\, (x \in f(x))$

**Solution:** In order to make $f$ one-to-one need to choose 3 *different* elements from $\mathcal{P}(A)$ (i.e., subsets of $A$) as the outputs of $f(a), f(b)$ and $f(c)$.

In order to satisfy the 2nd condition, for $f(a)$ we need to choose a subset of $A$ containing $a$. There are three such subsets of $A$: $\{a\}, \{a, b\}, \{a, b, c, \}$. In order to define $f$, we need choose one of them. Let's say we choose $\{a\}$, i.e., we set $f(a) = \{a\}$. Then for $f(b)$ choose a subset of $A$ containing $b$, and for $f(c)$ choose a subset of $A$ containing $c$. A natural set of choices is

$f(a) = \{a\}$

$f(b) = \{b\}$

$f(c) = \{c\}$

But there are other correct solutions, e.g.,

$f(a) = \{a, b\}$ $f(b) = \{b, c\}$ $f(c) = \{a, b, c\}$

An interesting question is: how many different correct solutions are there, i.e., how many different such functions $f : A \to \mathcal{P}(A)$ satisfying those two properties?