2. Solving Systems of Linear Equations in Three Variables

To solve a system involving three variables, the goal is to eliminate one variable. This reduces the system to two equations in two variables. One strategy for eliminating a variable is to pair up the original equations two at a time.

PROCEDURE Solving a System of Three Linear Equations in Three Variables

Step 1 Write each equation in standard form $Ax + By + Cz = D$.

Step 2 Choose a pair of equations, and eliminate one of the variables by using the addition method.

Step 3 Choose a different pair of equations and eliminate the *same* variable.

Step 4 Once steps 2 and 3 are complete, you should have two equations in two variables. Solve this system by using the methods from Sections 3.2 and 3.3.

Step 5 Substitute the values of the variables found in step 4 into any of the three original equations that contain the third variable. Solve for the third variable.

Step 6 Check the ordered triple in each of the original equations. Then write the solution as an ordered triple within set notation.

Example 1 Solving a System of Linear Equations in Three Variables

Solve the system.

$$\begin{aligned} 2x + y - 3z &= -7 \\ 3x - 2y + z &= 11 \\ -2x - 3y - 2z &= 3 \end{aligned}$$

Solution:

$$\begin{aligned} \boxed{A} \quad 2x + y - 3z &= -7 \\ \boxed{B} \quad 3x - 2y + z &= 11 \\ \boxed{C} \quad -2x - 3y - 2z &= 3 \end{aligned}$$

Step 1: The equations are already in standard form.

- It is often helpful to label the equations.
- The y variable can be easily eliminated from equations $\boxed{A}$ and $\boxed{B}$ and from equations $\boxed{A}$ and $\boxed{C}$. This is accomplished by creating opposite coefficients for the y terms and then adding the equations.

TIP: It is important to note that in steps 2 and 3, the *same* variable is eliminated.

Step 2: Eliminate the y variable from equations $\boxed{A}$ and $\boxed{B}$.

$$\begin{aligned} \boxed{A} \quad 2x + y - 3z = -7 &\xrightarrow{\text{Multiply by 2.}} 4x + 2y - 6z = -14 \\ \boxed{B} \quad 3x - 2y + z = 11 &\longrightarrow \underline{3x - 2y + z = 11} \\ & \qquad 7x \qquad - 5z = -3 \quad \boxed{D} \end{aligned}$$

Step 3: Eliminate the y variable again, this time from equations $\boxed{A}$ and $\boxed{C}$.

$$\begin{array}{ll} \boxed{A} \quad 2x + y - 3z = -7 \xrightarrow{\text{Multiply by 3.}} & 6x + 3y - 9z = -21 \\ \boxed{C} \quad -2x - 3y - 2z = 3 \longrightarrow & \underline{-2x - 3y - 2z = 3} \\ & 4x \qquad - 11z = -18 \quad \boxed{E} \end{array}$$

Step 4: Now equations $\boxed{D}$ and $\boxed{E}$ can be paired up to form a linear system in two variables. Solve this system.

$$\begin{array}{ll} \boxed{D} \quad 7x - 5z = -3 \xrightarrow{\text{Multiply by } -4.} & -28x + 20z = 12 \\ \boxed{E} \quad 4x - 11z = -18 \xrightarrow[\text{Multiply by 7.}]{} & \underline{28x - 77z = -126} \\ & -57z = -114 \\ & z = 2 \end{array}$$

Once one variable has been found, substitute this value into either equation in the two-variable system, that is, either equation $\boxed{D}$ or $\boxed{E}$.

$$\begin{aligned} \boxed{D} \quad 7x - 5z &= -3 \\ 7x - 5(2) &= -3 \qquad \text{Substitute } z = 2 \text{ into equation } \boxed{D}. \\ 7x - 10 &= -3 \\ 7x &= 7 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} \boxed{A} \quad 2x + y - 3z &= -7 \\ 2(1) + y - 3(2) &= -7 \\ 2 + y - 6 &= -7 \\ y - 4 &= -7 \\ y &= -3 \end{aligned}$$

Step 5: Now that two variables are known, substitute these values (x and z) into any of the original three equations to find the remaining variable y. Substitute $x = 1$ and $z = 2$ into equation $\boxed{A}$.

The solution set is $\{(1, -3, 2)\}$.

Step 6: Check the ordered triple in the three original equations.

<u>Check:</u>

$$\begin{array}{ll} 2x + y - 3z = -7 \rightarrow & 2(1) + (-3) - 3(2) \stackrel{?}{=} -7 \checkmark \text{ True} \\ 3x - 2y + z = 11 \rightarrow & 3(1) - 2(-3) + (2) \stackrel{?}{=} 11 \checkmark \text{ True} \\ -2x - 3y - 2z = 3 \rightarrow & -2(1) - 3(-3) - 2(2) \stackrel{?}{=} 3 \checkmark \text{ True} \end{array}$$

Skill Practice Solve the system.

1. $$\begin{aligned} x + 2y + z &= 1 \\ 3x - y + 2z &= 13 \\ 2x + 3y - z &= -8 \end{aligned}$$

Answer

1. $\{(1, -2, 4)\}$