College Algebra and Trigonometry

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$$\pi^{\ln\left(\sin^2\theta + \cos^2\theta\right)} = ?$$

Preface

This textbook is based on the course "College Algebra and Trigonometry" taught at New York City College of Technology, CUNY. It is designed to prepare students for the precalculus class at the next level.

The book is meant to be concise, while at the same time including all the material taught by mathematics department of NYCCT. Special attention is paid to present the material in a motivated and intuitively clear manner.

The book is divided into three parts and 25 sessions based on the total of 30 sessions in one semester, leaving remaining 5 sessions for 3 in-class exams, one review for final exam, and one final exam.

Each session ends with exercises. Most exercises are designed in pairs with consecutive odd and even numbers in such a way that exercises are similar in each pair. A recommended approach is for instructors to work out the examples in the lecture part of the session during class, then allow students to work on even exercises during class and assign the odd exercises for homework (or vice versa). The textbook also contains answers to almost all exercises.

Some sessions contain challenge problems. In sessions 4, 5, 8, 10, 13 and 23, challenge problems are given in parametric form, so they also can be used by instructors to generate problems for quizzes and exams by replacing parameters with numbers.

I would like to thank my colleagues from the Mathematics Department. My special thanks to Thomas Tradler and Holly Carley for a careful reading of the first part of the book and many valuable comments and suggestion. I am grateful to Roy Berglund for a detailed review of the text and many useful tips. I thank Satyanand Singh and Joel Greenstein for suggesting for improving the text.

Also, I would like to thank my brother, Dr. Leonid Rozenblum, for numerous corrections and recommendations.

Alexander Rozenblyum
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The textbook was updated on January 2022. In particular, some bugs were fixed, more problems were added, including challenges (Ch. 2, 4, 8, 10, 14, 15, 16, 18, 19, 20, 21, 23). Some challenges problems can be used as small student projects.

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Another update was done during Spring and Summer 2022, as well as on January 2023.

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Contents

Part I

Rational and Irrational Expressions and Equations

Session 1

Systems of Three Linear Equations in Three Variables

We assume that readers are already familiar with linear equations in one variable and with systems of two linear equations in two variables. Recall that linear equations in one variable x can be written as $ax + b = c$, and systems of two linear equations in two variables x and y can be written as

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}.$$

In this session, we consider systems of three linear equations in three variables.

The general form of such systems is this

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}.$$

Here x, y, and z are variables (unknown values). All other letters are given numbers. Numbers that are written next to variables (labeled with letters a, b and c) are called the **coefficients** of the system. The above system has nine coefficients. A solution of the system is a triple (X, Y, Z) that satisfies the system (i.e., makes each equation a true statement after substituting these numerical values for variables (x, y, z)).

Note. Keep in mind that a triple (X, Y, Z) represents one solution, not three.

When solving equations, we often deal with the movement of terms from one side of the equation to another. Technically, this action can be performed directly by moving the terms and changing their signs (see Example 1.0 below) or indirectly by adding or subtracting the same terms on both sides of the equation. We call the direct method the **Moving** method, and the indirect method the **Adding** method. As an example, let's compare these methods when solving the equation $2x - 1 = x + 5$.

Example 1.0. Solve the equation $2x - 1 = x + 5$ by both methods: adding and moving.

To solve the equation, we collect the terms with unknown x on the left side of the equation, and the numerical terms on the right side.

Adding Method	Moving Method
Write +1 underneath both sides of the equation: $$2x - 1 = x + 5$$ $$+ 1 \qquad + 1$$ Cancel –1 and 1 on the left side and add 5 and 1 on the right side: $$2x = x + 6$$ Write $-x$ underneath both sides: $$2x = x + 6$$ $$-x \quad -x$$ Subtract terms on the left side and cancel x on the right side: $$x = 6.$$	Move (–1) to the right side (and change it to +1), and move x to the left side (and change it to $-x$): $$2x - x = 5 + 1$$ Subtract terms on the left side and add terms on the right side: $$x = 6.$$

As you can see, the Moving method is simpler. In this textbook, we will most often use it.

There are different methods of solving systems of linear equations with any number of equations and any number of variables. Here we consider the **elimination method**.

Using this method, we can follow these steps:

1) Eliminate one of the variables of the give system. To do this, first decide which variable to eliminate. Theoretically, you may eliminate either one. Try to eliminate variable with the smaller coefficients. To eliminate, take any pair of equations of the system and proceed in the same way as for 2 x 2 system: multiply equations by appropriate numbers and then add them up. As a result, you will get an equation with the two remaining variables.

2) Take another pair of equations and eliminate the same variable. Do not eliminate another variable, stay with the variable that you choose. As a result, you will get another equation with the same two remaining variables as in step 1.

3) Combine two equations that you got in steps 1 and 2 into 2 x 2 system and solve it. You can use elimination method again. As a result, you will get numerical values of two variables.

4) Substitute the above values into one of the equations of the original system and solve it for the third variable.

Theoretically, there are three possibilities regarding the number of solutions of the linear system: it may have

1) One solution (so, one triple). We also say that the system has **unique** solution.

2) No solutions at all. Such system is called **inconsistent**.

3) Infinite many solutions. Such system is called **dependent**.

Examples below demonstrate the elimination method.

Example 1.1. Solve the system

$$\begin{cases} 3x - y + 2z = -3 \\ 2x + 4y - 5z = 1 \\ -8x + 3y + 3z = 17 \end{cases}.$$

Solution.

1) We have many options for choosing a variable to eliminate. In fact, we can eliminate any. Let's eliminate y. As the first pair of equations, we take the first and the second:

$$\begin{cases} 3x - y + 2z = -3 \\ 2x + 4y - 5z = 1 \end{cases}$$

To eliminate y, we want the coefficients for y in both equations to be equal in absolute values but have the opposite signs. In this case, if we add the equations, variable y will be cancelled (eliminated). To get this case, it's enough to multiply the first equation by 4.

Note. Multiplication an equation by a number means multiplying **all** the terms (on the left and right sides) of the equation by that number. After multiplying the first equation by 4, we get

$$\begin{cases} 12x - 4y + 8z = -12 \\ 2x + 4y - 5z = 1 \end{cases}$$

Now add these equations and y is eliminated:

$$12x + 2x + 8z - 5z = -12 + 1, \text{ or } 14x + 3z = -11.$$

2) As the second pair of equations, we take the first and third:

$$3 \begin{cases} 3x - y + 2z = -3 \\ -8x + 3y + 3z = 17 \end{cases}$$

Number 3 outside the braces means that we intend to multiply the first equation by 3:

$$\begin{cases} 9x - 3y + 6z = -9 \\ -8x + 3y + 3z = 17 \end{cases}$$

Add these equations to eliminate y: $9x - 8x + 6z + 3z = -9 + 17$, or $x + 9z = 8$.

3) Combine the resulting equations from steps 1 and 2 into one system:

$$\begin{cases} 14x + 3z = -11 \\ x + 9z = 8 \end{cases}.$$

To solve this system, we use the elimination method again:

$$\left. \begin{cases} 14x + 3z = -11 \\ x + 9z = 8 \end{cases} \right\} \Rightarrow \begin{cases} 14x \quad + 3z = -11 \\ -14\ x - 126z = -112 \end{cases}$$

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$$3z - 126z = -11 - 112 \ \Rightarrow \ -123z = -123 \ \Rightarrow \ z = 1.$$

At this point we found the variable $z = 1$. Substitute it into the second equation in step 3) and solve for x:

$$x + 9 \cdot 1 = 8 \ \Rightarrow \ x = -1.$$

4) Substitute the values $x = -1$ and $z = 1$ into the first equations of the original system, and solve it for y:

$$3 \cdot (-1) - y + 2 \cdot 1 = -3, \ \ -3 - y + 2 = -3, \ \ -y = -3 + 3 - 2, \ \ -y = -2, \ \ y = 2.$$

Final answer: the system has one (**unique**) solution

$$x = -1, \ y = 2, \ z = 1, \text{ or as a triple } (-1, 2, 1).$$

Example 1.2. Solve the system

$$\begin{cases} x - 2y + 4z = 5 \\ 2x + 3y - z = 1 \\ 4x - y + 7z = 7 \end{cases}.$$

Solution.

1) Let's eliminate z. As the first pair of equations, we take the first and the second:

$$\left. \begin{cases} x - 2y + 4z = 5 \\ 2x + 3y - z = 1 \end{cases} \right\} \Rightarrow \begin{cases} x - 2y + 4z = 5 \\ 8x + 12y - 4z = 4 \end{cases}$$

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Add the last equations to eliminate z: $x + 8x - 2y + 12y = 5 + 4, \ \ 9x + 10y = 9.$

2) As the second pair of equations, we take the second and the third:

$$\left. \begin{cases} 2x + 3y - z = 1 \\ 4x - y + 7z = 7 \end{cases} \right\} \Rightarrow \begin{cases} 14x + 21y - 7z = 7 \\ 4x \ - \ y + 7z = 7 \end{cases}$$

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Add the last equations: $14x + 4x + 21y - y = 7 + 7, \ \ 18x + 20y = 14, \ \ 9x + 10y = 7.$

3) Combine the resulting equations from steps 1 and 2 into one system:

$$\begin{cases} 9x + 10y = 9 \\ 9x + 10y = 7 \end{cases}$$

Notice that the left sides of both equations are the same, but the right sides are different. Therefore, this system does not have solutions, so the system is **inconsistent**.

Final answer: the system does not have solutions. We can also say that the solution set is an empty set (the symbol for empty set is $\varnothing$).

Example 1.3. Solve the system

$$\begin{cases} 2x - 4y + 3z = 5 \\ 8x - 6y + 5z = 7 \\ x + 3y - 2z = -4 \end{cases}.$$

Solution.

1) Let's eliminate x. As the first pair of equations, we take the first and the second:

$$\begin{matrix} & \begin{cases} 2x - 4y + 3z = 5 \\ x + 3y - 2z = -4 \end{cases} \\ -2 & \end{matrix} \Rightarrow \begin{cases} 2x - 4y + 3z = 5 \\ -2x - 6y + 4z = 8 \end{cases}$$

Add the last equations to eliminate x: $-4y - 6y + 3z + 4z = 5 + 8,\ -10y + 7z = 13$.

2) As the second pair of equations, we take the second and the third:

$$\begin{matrix} & \begin{cases} 8x - 6y + 5z = 7 \\ x + 3y - 2z = -4 \end{cases} \\ -8 & \end{matrix} \Rightarrow \begin{cases} 8x - 6y + 5z = 7 \\ -8x - 24y + 16z = 32 \end{cases}$$

Add the last equations: $-6y - 24y + 5z + 16z = 7 + 32,\ -30y + 21z = 39$,

$-10y + 7z = 13$.

3) Combine the resulting equations from steps 1 and 2 into one system:

$$\begin{cases} -10y + 7z = 13 \\ -10y + 7z = 13 \end{cases}$$

Notice that both equations coincide. So, actually, we have only one equation. In this case we cannot find the values of y and z uniquely. Indeed, we can assign any numerical value to one of the variables y or z, say to z. Then we can solve the above equation for y. Since there are infinite values of z to choose from, we get an infinite number of pairs (y, z) which are solutions of the above 2 x 2 system. It means that the original system has infinitely many solutions. Substituting y and z into any of the original equations, we can find x. Finally, we will get infinitely many triples (X, Y, Z). So, the system is **dependent**.

We come up to an interesting question, how to describe an infinite set of all solutions of the system. Of course, we cannot create an infinite list of them. Instead, we can use the **parametric** form to describe the solution set. It means the following. Let's solve the above equation $-10y + 7z = 13$ for y in terms of z:

$$-10y + 7z = 13 \Rightarrow -10y = 13 - 7z \Rightarrow y = -\frac{13}{10} + \frac{7}{10}z.$$

Here the variable z may take any values, and we call it the **free parameter**. Let's denote this parameter by the letter t: $z = t$. Then, $y = -\frac{13}{10} + \frac{7}{10}t$. Now, we can express the

variables x in terms of the parameter t by substituting expressions for y and z into any equation of the original system. Let's substitute expressions for y and z into the third equation (in which the coefficient for x is 1) and solve for x:

$$x + 3\left(-\frac{13}{10} + \frac{7}{10}t\right) - 2t = -4 \;\Rightarrow\; x - \frac{39}{10} + \frac{21}{10}t - 2t = -4,$$

$$x = \frac{39}{10} - \frac{21}{10}t + 2t - 4 = \frac{39 - 21t + 20t - 40}{10} = \frac{-1-t}{10}.$$

We can also write x as $x = -\dfrac{1}{10} - \dfrac{1}{10}t$. Now we have described all unknowns in parametric form:

$$x = -\frac{1}{10} - \frac{1}{10}t, \quad y = -\frac{13}{10} - \frac{7}{10}t, \quad z = t.$$

Here t is a parameter that takes any numerical value.

Note. We can get the specific (particular) numerical solutions of the original system from the above parametric representation by assigning any specific number to the parameter t. For example, if we put $t = 0$, we get the particular solution $x = -\dfrac{1}{10}, \; y = -\dfrac{13}{10}, \; z = 0$.

Exercises 1

In exercises 1.1 and 1.2, solve the system of equations.

If the system is inconsistent, state that.

If the system is dependent, state that.

1.1.

a) $\begin{cases} 3x + 6y - 4z = 1 \\ 5y + 2z = 8 \\ 6z = -6 \end{cases}$

b) $\begin{cases} 2x - 3y + z = -10 \\ 5x + 2y - 3z = 1 \\ -3x + 4y + 5z = -5 \end{cases}$

c) $\begin{cases} 3x + 2y + z = 1 \\ 11x + 10y + 9z = 5 \\ x + 2y + 3z = 1 \end{cases}$

d) $\begin{cases} 4x - 2y + 3z = 7 \\ 6x + y - 2z = -4 \\ -5x + 3y + 4z = -2 \end{cases}$

e) $\begin{cases} 5x + 3y - 8z = 6 \\ x + y - 2z = 1 \\ 3x + 2y - 5z = 4 \end{cases}$

1.2.

a) $\begin{cases} 4x + 3y - 2z = 4 \\ 2y - 5z = -19 \\ 3z = 9 \end{cases}$

b) $\begin{cases} 4x + 2y - 3z = 10 \\ -x + 2y + 5z = -21 \\ 8x + 7y - 5z = 6 \end{cases}$

c) $\begin{cases} x - 3y + z = 4 \\ 3x + 2y + z = 3 \\ 6x + 4y + 2z = -1 \end{cases}$

d) $\begin{cases} 5x - 4y + 2z = 9 \\ 2x + y - 4z = -6 \\ 8x - 3y - 3z = 2 \end{cases}$

e) $\begin{cases} 3x + 2y + z = 4 \\ x - y + 2z = 3 \\ 4x + y + 3z = 7 \end{cases}$

Challenge Problem

1.3. For the systems in exercises 1.1 and 1.2, that are dependent,

   1) Describe solutions in parametric form.
   2) Find a particular solution.

   (Answers may vary).

Session 1A

Determinants and Cramer's Rule

In the previous session, we solved systems of three linear equations by the elimination method. This method requires some specific operations performed on the equations of given systems. In this session, we consider **formulas** that allow us to explicitly calculate solutions by directly substituting the coefficients of the equations into these formulas, instead of manipulating the equations. Such formulas are called the Cramer's rule named after Gabriel Cramer (1704 – 1752), a Swiss mathematician. Cramer's rule is not efficient for systems with many equations, and it is not used in practice. However, it is easy enough to use for systems with two and three equations that we consider here. Cramer's rule is especially convenient when the coefficients of given system are integers, but the solutions are fractions.

Case of a system with two equations

We derive Cramer's rule for the system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

First, we solve this system by elimination method. Let's eliminate the variable y by multiplying the first equation by e, the second equation by $–b$, and adding the resulting equations:

$$\begin{matrix} e \\ -b \end{matrix} \begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} aex + bey = ce \\ -bdx - bey = -bf \end{cases}$$

Add the equations on the right, and solve for x:

$aex - bdx = ce - bf \Rightarrow (ae - bd)x = ce - bf$. So, if $ae - bd \neq 0$,

$$x = \frac{ce - fb}{ae - bd}.$$

In similar way we can find y by eliminating x:

$$\begin{matrix} -d \\ a \end{matrix} \begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow \begin{cases} -adx - bdy = -cd \\ adx + aey = af \end{cases}$$

Add the equations on the right, and solve for y:

$-bdy + aey = -cd + af \Rightarrow (ae - bd)y = af - cd$. So, if $ae - bd \neq 0$,

$$y = \frac{af - cd}{ae - bd}.$$

We have come up with the following general formulas for the solutions of a system of two linear equations with two variables:

$$x = \frac{ce - fb}{ae - bd}, \quad y = \frac{af - cd}{ae - bd}, \text{ (when } ae - bd \neq 0 \text{)}.$$

Notice that the denominators of both fractions are the same, and the structure of the numerators looks similar to that of the denominators. Cramer's rule represents these formulas in terms of a special number that is called the **determinant**. The determinant is defined by four numbers, say k, l, m, and n. Here is the notation and the definition of the determinant:

$$\begin{vmatrix} k & l \\ m & n \end{vmatrix} = kn - ml.$$

We call this a 2×2 determinant. As you can see, to calculate the determinant, we take the product along the main diagonal (from left top corner to right bottom corner, so we multiply k by n) minus the product along the minor diagonal (from left bottom corner to right top corner, so we multiply m by l).

If you return to the above formulas for x and y, you may notice that their numerators and denominators can be written in terms of determinants. We thus derived the following rule, which is called the **Cramer's rule:**

To solve the system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

we proceed in the following three steps:

1) Calculate the determinant D which is called the determinant of the system:

$$D = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd.$$

   Notice that the coefficients c and f from the right side of the system are not used in this determinant. It consists of the coefficients for x and y only.

2) Calculate another two determinants, D_x and D_y:

$$D_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf, \quad D_y = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd.$$

   Notice that the determinant D_x is obtained from D by replacing its first column with the column of the coefficients c and f. Similarly, the determinant D_y is obtained from D by replacing its second column with the column of "free" coefficients.

3) If $D \neq 0$, calculate the solution of the system by the formulas

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}.$$

Note. As you see, the denominator in these fractions is the determinant D. Therefore, these formulas make sense only if $D \neq 0$. If $D = 0$, then the system does not have a unique solution. Instead, it either does not have solutions at all (the system is inconsistent), or it has infinitely many solutions (the system is dependent). To detect which case we have, we should check D_x (or D_y) for zero. If $D_x \neq 0$, then there are no solutions. If $D_x = 0$, then the system has an infinite number of solutions. (If $D = 0$, then both D_x and D_y are equal or not equal to zero simultaneously, see exercise 1A.7).

Example 1A.1. Solve the following system using the Cramer's rule.

$$\begin{cases} 7x - 2y = 4 \\ 5x + 3y = 7 \end{cases}.$$

Solution.

1) Calculate the determinant D of the system:

$$D = \begin{vmatrix} 7 & -2 \\ 5 & 3 \end{vmatrix} = 7 \cdot 3 - 5 \cdot (-2) = 21 + 10 = 31.$$

2) Calculate the determinants D_x and D_y:

$$D_x = \begin{vmatrix} 4 & -2 \\ 7 & 3 \end{vmatrix} = 4 \cdot 3 - 7 \cdot (-2) = 12 + 14 = 26,$$

$$D_y = \begin{vmatrix} 7 & 4 \\ 5 & 7 \end{vmatrix} = 7 \cdot 7 - 5 \cdot 4 = 49 - 20 = 29.$$

3) Write the solution of the system:

$$x = \frac{D_x}{D} = \frac{26}{31}, \quad y = \frac{D_y}{D} = \frac{29}{31}.$$

Final answer: $x = \dfrac{26}{31}, \ y = \dfrac{29}{31}$, or, as a pair, $\left(\dfrac{26}{31}, \ \dfrac{29}{31} \right)$.

Case of a system with three equations

We will describe the Cramer's rule for the system

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}.$$

Similar to systems with two equations, the solutions of this system can also be represented in terms of determinants as ratios of determinants D_x, D_y, and D_z corresponding to the variables x, y, and z, to the common determinant D of the system. We now describe how to find these determinants.

We will not derive these formulas in this text, but just provide the final result. The determinant D of the above system is denoted by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This is a 3×3 determinant constructed from the coefficients of the system. There are several methods to calculate it. We only consider here one method: a direct calculation.

Direct calculation method. Here is the formula for the determinant D:

$$D = a_1 b_2 c_3 + b_1 c_2 a_3 + a_2 b_3 c_1$$
$$- c_1 b_2 a_3 - b_1 a_2 c_3 - c_2 b_3 a_1 .$$

You can consider this formula as a definition of the determinant. It seems to be difficult to memorize such a formula. Notice that it contains six terms: three terms with the plus sign, and another three with the minus sign. Here is one of the possible ways to remember this formula. Extend (double) the determinant D to the following table:

$$\begin{bmatrix} a_1 & b_1 & c_1 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 & c_3 \end{bmatrix}.$$

To get the three terms of the determinant with the plus sign, calculate products along the main diagonal a_1, b_2, c_3, and two parallel diagonals b_1, c_2, a_3 and c_1, a_2, b_3.

To get the three terms with the minus sign, calculate products along the minor diagonal c_1, b_2, a_3, and two parallel diagonals a_1, c_2, b_3 and b_1, a_2, c_3. Symbolically, we multiply coefficients along the lines

Terms with the plus sign   Terms with the plus sign

Note. The last column of the above table is not used, so it is not necessary to write it.

Example 1A.2. Calculate the following determinant by direct calculation

$$D = \begin{vmatrix} 5 & -6 & -2 \\ 3 & 2 & -4 \\ 2 & 0 & 3 \end{vmatrix}.$$

Solution. Construct the extended table (we dropped the last column)

$$\begin{bmatrix} 5 & -6 & -2 & 5 & -6 \\ 3 & 2 & -4 & 3 & 2 \\ 2 & 0 & 3 & 2 & 0 \end{bmatrix}$$

We have

$$D = 5 \cdot 2 \cdot 3 + (-6) \cdot (-4) \cdot 2 + (-2) \cdot 3 \cdot 0$$

$$-2 \cdot 2 \cdot (-2) - 0 \cdot (-4) \cdot 5 - 3 \cdot 3 \cdot (-6) = 30 + 48 + 8 + 54 = 140.$$

Now, we are ready to describe the Cramer's rule for the system

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}.$$

To solve the above system we proceed in the following three steps:

1) Calculate the determinant D of this system:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2) Calculate the three other determinants, D_x, D_y and D_z that correspond to variables x, y, and z. These determinants are constructed by replacing corresponding columns of the determinant D with the column from the right side of the system:

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

3) If $D \neq 0$, calculate the solution of the system via the formulas

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}.$$

Note. Similar to the case of a system with two variables, if $D = 0$, then either there is no solution, or there is an infinite number of solutions.

Example 1A.3. Solve the following system using the Cramer's rule.

$$\begin{cases} 5x - 6y - 2z = 7 \\ 3x + 2y - 4z = -8 \\ 2x + 3z = 5 \end{cases}$$

Solution.

1) The determinant D of the system is $D = \begin{vmatrix} 5 & -6 & -2 \\ 3 & 2 & -4 \\ 2 & 0 & 3 \end{vmatrix}$.

This is exactly the same determinant as in example 1A.2, so $D = 140$.

2) Calculate the determinants D_x, D_y and D_z:

$$D_x = \begin{vmatrix} 7 & -6 & -2 \\ -8 & 2 & -4 \\ 5 & 0 & 3 \end{vmatrix}, \quad D_y = \begin{vmatrix} 5 & 7 & -2 \\ 3 & -8 & -4 \\ 2 & 5 & 3 \end{vmatrix}, \quad D_z = \begin{vmatrix} 5 & -6 & 7 \\ 3 & 2 & -8 \\ 2 & 0 & 5 \end{vmatrix}.$$

For determinate D_x, construct the extended table

$$\begin{bmatrix} 7 & -6 & -2 & 7 & -6 \\ -8 & 2 & -4 & -8 & 2 \\ 5 & 0 & 3 & 5 & 0 \end{bmatrix}.$$

$$D_x = 7 \cdot 2 \cdot 3 + (-6) \cdot (-4) \cdot 5 + (-2) \cdot (-8) \cdot 0$$
$$- 5 \cdot 2 \cdot (-2) - 0 \cdot (-4) \cdot 7 - 3 \cdot (-8) \cdot (-6) = 38$$

For determinate D_y, construct the extended table

$$\begin{bmatrix} 5 & 7 & -2 & 5 & 7 \\ 3 & -8 & -4 & 3 & -8 \\ 2 & 5 & 3 & 2 & 5 \end{bmatrix}.$$

$$D_y = 5 \cdot (-8) \cdot 3 + 7 \cdot (-4) \cdot 2 + (-2) \cdot 3 \cdot 5$$
$$- 2 \cdot (-8) \cdot (-2) - 5 \cdot (-4) \cdot 5 - 3 \cdot 3 \cdot 7 = -201$$

For determinate D_z, construct the extended table

$$\begin{bmatrix} 5 & -6 & 7 & 5 & -6 \\ 3 & 2 & -8 & 3 & 2 \\ 2 & 0 & 5 & 2 & 0 \end{bmatrix}.$$

$$D_z = 5 \cdot 2 \cdot 5 + (-6) \cdot (-8) \cdot 2 + 7 \cdot 3 \cdot 0$$
$$- 2 \cdot 2 \cdot 7 - 0 \cdot (-8) \cdot 5 - 5 \cdot 3 \cdot (-6) = 208$$

3) Calculate the solutions x, y, and z of the system

$$x = \frac{D_x}{D} = \frac{38}{140} = \frac{19}{70}, \quad y = \frac{D_y}{D} = -\frac{201}{140}, \quad z = \frac{D_z}{D} = \frac{208}{140} = \frac{52}{35}.$$

Final answer: $x = \dfrac{19}{70}$, $y = -\dfrac{201}{140}$, $z = \dfrac{52}{35}$, or, as a triple, $\left(\dfrac{19}{70}, \ -\dfrac{201}{140}, \ \dfrac{52}{35} \right)$.

Exercises 1A

In exercises 1A.1 and 1A.2, solve the systems of equations using Cramer's rule.

1A.1. $\begin{cases} -3x+5y=4 \\ 7x-6y=8 \end{cases}$

1A.2. $\begin{cases} 2x-3y=7 \\ 5x+4y=9 \end{cases}$

In exercises 1A.3 and 1A.4, calculate the determinants.

1A.3. $D = \begin{vmatrix} 3 & -2 & 1 \\ -5 & 4 & -3 \\ 2 & 1 & 7 \end{vmatrix}$

1A.4. $D = \begin{vmatrix} 1 & 6 & -7 \\ 7 & -5 & 3 \\ 4 & -3 & 2 \end{vmatrix}$

In exercises 1A.5 and 1A.6, use the results of exercises 1A.3 and 1A.4 respectively to solve the systems of equations using Cramer's rule.

1A.5. $\begin{cases} 3x-2y+z=1 \\ -5x+4y-3z=0 \\ 2x+y+7z=2 \end{cases}$

1A.6. $\begin{cases} x+6y-7z=2 \\ 7x-5y+3z=-1 \\ 4x-3y+2z=0 \end{cases}$

Challenge Problem

1A.7. As we described for the 2×2 system $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$, determinants D, D_x and D_y are defined by the formulas

$$D = ae-bd, \quad D_x = ce-bf, \text{ and } D_y = af-cd.$$

Let $D=0$ and $D_x=0$. Show that if b or e is not zero, then $D_y=0$.

Give an example when both b and e are zeros, and this statement is wrong (i.e. $D=0$, $D_x=0$, but $D_y \neq 0$).

Session 2

Quadratic Equations: Factored Form

When solving a linear equation $ax + b = 0$ with $a \neq 0$, we always get a unique solution $x = -\dfrac{b}{a}$. In some cases, problems arise leading to more complicated equations that may have more than one solution.

Example 2.0. Suppose you need to measure a piece of land in the shape of rectangle, having a given area A and a given perimeter P (length of fence). What are the sides of this rectangle?

Solution (equation only). Let's denote the sides of the rectangle by variables x and y. Then $x \cdot y = A$ (area), and $2x + 2y = P$ (perimeter). We can solve the last equation for y:

$2y = P - 2x$, and $y = \dfrac{P - 2x}{2}$. If we substitute this expression for y into the first equation $x \cdot y = A$, we will get

$$x \cdot y = x \cdot \frac{P - 2x}{2} = A \implies x(P - 2x) = 2A \implies xP - 2x^2 = 2A \implies 2x^2 - Px + 2A = 0.$$

We come up with an equation that contains x^2. We read this expression as "x squared". By definition, $x^2 = x \cdot x$. We will study expressions like this in details in session 3.

Informally, equations that contain x^2 and are similar to the above, are called the **quadratic equations** (as opposed to linear equations: $ax + b = 0$).

More precisely, an equation for the variable x is called the **quadratic** equation, if it can be written in the form

$$ax^2 + bx + c = 0, \ a \neq 0.$$

This form is called the **standard form**. Here a, b, and c are constant numbers which are called the coefficients: a is called the leading, and c is called free (free of x) coefficient.

Left side of this equation is called the **quadratic trinomial**. Roots (solutions) of the quadratic equation are also called roots of the quadratic trinomial.

Note. If we omit the restriction $a \neq 0$, the above equation will not necessarily be a quadratic: when $a = 0$, the equation becomes linear $bx + c = 0$. Therefore, when dealing with quadratic equations, we will always assume that the leading coefficient $a \neq 0$. Coefficients b and c may be any real numbers, including 0. Also notice that in the standard form the right side of the equation is equal to zero.

The quadratic equation can be given in various forms, but if necessary, it can always be presented in a standard form.

Example 2.1. Write the following equations as quadratic equations in standard form. Identify the coefficients a, b, and c.

1) $(2x - 1)(x + 5) = 0$.

2) $(3x + 2)^2 = 5$.

Solution. In both equations we just need to distribute, combine like terms, and bring all terms from the right side to the left (if needed).

1) $(2x - 1)(x + 5) = 2x^2 + 10x - x - 5 = 2x^2 + 9x - 5$.

    We get the standard form

$$2x^2 + 9x - 5 = 0, \text{ so, } a = 2, b = 9, c = -5.$$

2) $(3x + 2)^2 = 5 \ \Rightarrow \ 9x^2 + 12x + 4 = 5 \ \Rightarrow \ 9x^2 + 12x + 4 - 5 = 0 \ \Rightarrow \ 9x^2 + 12x - 1 = 0$.

    We get the standard form

$$9x^2 + 12x - 1 = 0, \text{ so, } a = 9, b = 12, c = -1.$$

Notes.

1) In Example 2.1, 1), we say that the equation is written in the **factored form**.

2) In Example 2.1, 2), we say that the equation is written in the **squared form**. We will discuss this form in session 9.

3) In solving Example 2.1, 2), we used the following formula (square of the sum formula):

$$\boxed{\left(a + b\right)^2 = a^2 + 2ab + b^2}$$

Another useful formula is the square of the difference formula:

$$\boxed{\left(a - b\right)^2 = a^2 - 2ab + b^2}$$

4) We will also use the following formula which is called the difference of squares:

$$\boxed{a^2 - b^2 = (a - b)(a + b)}$$

This formula tells us how to factor the difference of two squares. Try to memorize the above three formulas.

When solving a quadratic equation, it is not necessary to always represent it in the standard form. In some cases, other forms may be preferable, as in Example 2.1: the factored form or the squared form. These are the forms in which the quadratic equation can be solved easier than in the standard form. Here we consider the factored form, and in session 9 – the squared form.

Factored Form of the Quadratic Equation

In general, the factored form of the quadratic equation looks like this:

$$(mx + n)(px + q) = 0.$$

The method to solve this equation is based on the following simple observation: if the product of two values A and B is zero, i.e. $A \cdot B = 0$, then at least one of them is zero: $A = 0$ and/or $B = 0$. Using this property (which is called the **zero-product property**), the equation $(mx + n)(px + q) = 0$ can be split into two linear equations:

$$mx + n = 0 \text{ and } px + q = 0,$$

which can be solved to get the solutions to the original equation.

Example 2.2. Solve the equation from Example 2.1, 1): $(2x - 1)(x + 5) = 0$.

Solution. Since this equation is written in factored form, using the zero-product property it can immediately be split into two equations: $2x - 1 = 0$ and $x + 5 = 0$. From the first equation, $x = \dfrac{1}{2}$, and from the second, $x = -5$. So, the original equation has two solutions:

$$x = \frac{1}{2} \text{ and } x = -5.$$

Many quadratic equations (but not all, if we use only rational numbers) can be solved by **factoring**. Using this method, we first represent the given equation in the factored form, and then split it into two linear equations like in Example 2.2. We consider separately two cases for quadratic equations: the case when the leading coefficient $a = 1$, and the case when $a \neq 1$.

Case: leading coefficient $a = 1$.

In this case the standard form of the equation is

$$x^2 + bx + c = 0.$$

Such an equation (when the leading coefficient is 1) is called the **reduced** equation. To factor, we need to represent the left side as a product of two linear expressions (two pairs of parentheses): $(x + p)(x + q) = 0$. Let's distribute the left side and combine like terms:

$$(x + p)(x + q) = x^2 + px + qx + pq = 0 \implies x^2 + (p + q)x + pq = 0.$$

If we compare the last equation with the original $x^2 + bx + c = 0$, we conclude that $p + q = b$ (coefficient for x), and $p \cdot q = c$ (free coefficient). This conclusion gives us an idea: to factor, we need to find two numbers p and q such that their sum is the middle coefficient b and the product is the last coefficient c.

Technically, to factor we can start with a template (skeleton) for the equation:

$$(x + \underline{\quad})(x + \underline{\quad}) = 0.$$

Then consider all possible ways to factor the last coefficient c, and select such factors whose sum is b. Replace the blanks with these numbers.

Example 2.3. Solve the quadratic equation $x^2 + 5x + 6 = 0$ by factoring.

Solution. Start with the template $(x + \_\_)(x + \_\_) = 0$. For the last coefficient 6, there are two ways to factor: $6 = 2 \times 3$ and $6 = 1 \times 6$ (we ignore here negative numbers). We select factors 2 and 3 since their sum is the middle coefficient 5. Replacing blanks with these numbers, we get the factored form
$$(x + 2)(x + 3) = 0.$$
Now, using the zero-product property, split it into two equations: $x + 2 = 0$ and $x + 3 = 0$. Solve each and get two solutions:
$$x = -2 \text{ and } x = -3.$$

Note. Keep in mind that in the factored form, the right side of the equation must be zero. For example, the equation $(x - 1)(x + 2) = 4$ is **NOT** written in factored form and cannot be split immediately into two linear equations.

Example 2.4. Solve the above equation $(x - 1)(x + 2) = 4$ by factoring.

Solution. To write this equation in factored form, we first represent it in standard form by distributing the left side and combining like terms
$$x^2 + 2x - x - 2 = 4, \text{ or } x^2 + x - 6 = 0.$$
Now to factor, we write the template $(x + \_\_)(x + \_\_) = 0$, and find two numbers such that the product is -6 and the sum is 1 (which is the coefficient for x). We can find 3 and -2. Substitute these numbers for the blanks and get the factored form $(x + 3)(x - 2) = 0$. Solving the equations $x + 3 = 0$ and $x - 2 = 0$, we get two solutions:
$$x = -3 \text{ and } x = 2.$$

Example 2.5. Solve the quadratic equation $x^2 + 7x = 0$ by factoring.

Solution. Here the coefficient $c = 0$. We can factor the left side just by taking x out of parentheses: $x(x + 7) = 0$. From here, $x = 0$ and $x + 7 = 0 \Rightarrow x = -7$. The final answer is:
$$x = 0 \text{ and } x = -7.$$

Example 2.6. Solve the quadratic equation $3x^2 - 48 = 0$ by factoring.

Solution. Here the leading coefficient is not 1, it is 3. Notice, however, that both coefficients 3 and -48 divisible by 3, and we can factor this 3 out: $3(x^2 - 16) = 0$. From here we conclude that the expression inside the parentheses must be zero, so we may drop the factor 3 and get an equation with the leading coefficient 1: $x^2 - 16 = 0$. Another way to get this equation is just to divide all terms of the original equation by 3. Now, to factor the left side of this equation, we may recognize it as the difference of two squares $a^2 - b^2$ with $a^2 = x^2$ and $b^2 = 16$. Using the formula $a^2 - b^2 = (a - b)(a + b)$, we write the equation

$x^2 - 16 = 0$ in the factored form $(x-4)(x+4) = 0$. Now it remains to solve the two equations $x - 4 = 0$ and $x + 4 = 0$, and get the final answer: $x = 4$ and $x = -4$. The final answer can also be written as $x = \pm 4$, meaning that we have combined both roots in one formula.

Case: leading coefficient $a \neq 1$.

We will present a method how to reduce the quadratic equation $ax^2 + bx + c = 0$ when $a \neq 1$ to the case $a = 1$ in such a way that if coefficients a, b and c are integers, then coefficients of the reduced equation are also integers. In some cases, it is easier to solve the reduced equation. The method is based on the following

Proposition. Let r be a root of the equation $x^2 + bx + ac = 0$. Then $\dfrac{r}{a}$ is a root of the equation $ax^2 + bx + c = 0$.

Proof. By definition of the root r, $r^2 + br + ac = 0$. Divide both sides by a:

$$\frac{r^2}{a} + \frac{br}{a} + \frac{ac}{a} = 0 \quad \Rightarrow \quad a\frac{r^2}{a^2} + b\frac{r}{a} + c = 0 \quad \Rightarrow \quad a\left(\frac{r}{a}\right)^2 + b\left(\frac{r}{a}\right) + c = 0.$$

The last equality means that $\dfrac{r}{a}$ is a root of the equation $ax^2 + bx + c = 0$. ∎

This proposition allows us to solve the equation $ax^2 + bx + c = 0$ in the following three steps:

1) Construct (temporary) a reduced equation $x^2 + bx + ac = 0$. In words: take away leading coefficient a from x^2 and multiply it by c.

2) Solve the above reduced equation. Let's its roots be r and s.

3) Divide both r and s by a to get the roots of the original equation $ax^2 + bx + c = 0$. The roots are $\dfrac{r}{a}$ and $\dfrac{s}{a}$.

Example 2.7. Solve the equation $6x^2 + 5x - 4 = 0$.

Solution. We use the above method.

1) Construct the reduced equation $x^2 + 5x - 6 \cdot 4 = 0$ or $x^2 + 5x - 24 = 0$.

2) Solve the reduced equation by factoring. To factor, we need to find two numbers such that the product is –24 and the sum is 5. These numbers are 8 and –3. The above reduced equation is factored as: $(x + 8)(x - 3) = 0$. Its roots are –8 and 3.

3) Divide both –8 and 3 by the leading coefficient 6 of the original equation to get its roots. The roots are $\dfrac{-8}{6} = -\dfrac{4}{3}$ and $\dfrac{3}{6} = \dfrac{1}{2}$.

For some problems, such as reducing rational expressions, we need to write a quadratic trinomial having leading coefficient a and roots r and s in the factored form. The result is this:

$$a(x - r)(x - s).$$

In the case when all coefficients of the quadratic trinomial are integers and roots r and s are rational numbers (fractions), the above factored form can be written in terms of integers.

Example 2.8. Factor the quadratic trinomial $6x^2 + 5x - 4$ from the left side of the equation in the problem 2.7.

Solution. Here, the leading coefficient is 6, and the roots are $-\frac{4}{3}$ and $\frac{1}{2}$. Therefore, the factored form is

$$6\left(x + \frac{4}{3}\right)\left(x - \frac{1}{2}\right).$$

To write this expression using integers, we factor coefficient 6 as 3×2, and multiply 3 by the first expression in parentheses, and 2 by the second. We get

$$6\left(x + \frac{4}{3}\right)\left(x - \frac{1}{2}\right) = 3\left(x + \frac{4}{3}\right)2\left(x - \frac{1}{2}\right) = (3x + 4)(2x - 1).$$

Exercises 2

In exercises 2.1 and 2.2, write the given equations as quadratic equations in standard form.

Identify the coefficients a, b, and c.

2.1 a) $(3x+7)(x-2)=0$

    b) $(4x-3)^2=6$

2.2. a) $(4x-3)(2x+1)=0$

     b) $(5x+2)^2=3$

In exercises 2.3 and 2.4, solve the given equations.

2.3. a) $(3x+7)(x-2)=0$

     b) $(6x-5)^2=0$

2.4. a) $(4x-3)(2x+1)=0$

     b) $(7x+4)^2=0$

In exercises 2.5 and 2.6, solve the given equations by factoring.

2.5. a) $3x^2+5x=0$

     b) $2x^2-32=0$

     c) $x^2+x-12=0$

     d) $x^2+10x+25=0$

     e) $(x+4)(x-3)=8$

2.6. a) $6x^2-7x=0$

     b) $5x^2-45=0$

     c) $x^2-2x-15=0$

     d) $x^2+12x+36=0$

     e) $(x+3)(x-5)=9$

In exercises 2.7 and 2.8, solve the given equations.

2.7. a) $8x^2-2x-3=0$

     b) $5x^2+12x=9$

     c) $9x^2+2=9x$

2.8. a) $6x^2+13x-5=0$

     b) $7x^2+3x=4$

     c) $12x^2-2=5x$

For exercises 2.9 and 2.10, write the equations from exercises 2.7 and 2.8, part b) and c), in the standard form. Then do exercises 2.9 and 2.10.

2.9. Factor trinomials from the problem 2.7 using integers.

2.10. Factor trinomials from the problem 2.8 using integers.

Challenge Problems

2.11. Let a and b be two positive numbers. Are the following expressions equal? If not, which one is bigger:
$$a^2 + b^2 \text{ or } (a + b)^2?$$

2.12. Prove that $\dfrac{a^2+b^2}{2} \geq ab$.

2.13. Calculate the following expression mentally without using a calculator or paper.

$$12345^2 - 12344^2$$

2.14. Develop a method to easily calculate the following expression without using a calculator.

$$\frac{12344 \cdot 12346 + 1}{12345}$$

2.15. Construct a quadratic equation with

a) roots 2 and –3

b) one root 4.

How many quadratic equations with the above roots can you construct?

2.16. Let A and B be two real numbers. What conclusion can you make about A and B from the following equations?

a) $A^2 + B^2 = 0$.

b) $\dfrac{AB}{A^2+B^2} = 0$.

c) $A^2 = B^2$.

2.17. Consider quadratic equation $x^2 + bx + c = 0$. Assume that it has two real roots p and q. What numbers (positive, negative, zero, or any) can coefficients b and c be, if?

a) $p + q > 0$

b) $p + q < 0$

c) $p + q = 0$

d) $pq > 0$

e) $pq < 0$

f) $pq = 0$.

2.18. Let the quadratic equation $x^2 - 2x + c = 0$ has the root of 4. Find the coefficient c.

2.19. Let the quadratic equation $x^2 + bx - 12 = 0$ has the root of 4. Find the coefficient b.

Session 3

Integer Exponents

Exponential Expressions with Positive Integer Powers

Let's recall the well-known notation of multiplication. Everybody knows that $3 \times 4 = 12$. But what does exactly multiplication mean? Why is the result 12? We got this result by **adding** number 3 to itself 4 times:

$$3 \times 4 = 3 + 3 + 3 + 3 = 12.$$

Multiplication means repetition with **addition**. It allows us to write the summation of a number with itself in a short, compact form.

There are cases when we need repetition with **multiplication**. In other words, we want to **multiply** a number by itself several times. For example, consider the product $3 \times 3 \times 3 \times 3$. It would be a good idea to invent a special notation, similar to the notation for multiplication, that allows to write such a product in a short form, using the number 3 only one time (which tells us the number we want to multiply by itself) and number 4 (which tells us how many times to multiply). We cannot use the notation 3×4 because it is already taken for multiplication to express repetition with summation. The following notation was invented to express repetition with multiplication: 3^4. This expression is called the **exponential expression**. So, by definition

$$3^4 = 3 \times 3 \times 3 \times 3.$$

Note. In some computer languages and calculators, to keep both numbers (3 and 4) on one line, the notation 3^4 is used.

In similar way, we can define the exponential expression in general form:

Definition. For arbitrary number a and arbitrary positive integer n the exponential expression a^n is defined by the formula:

$$a^n = a \times a \times ... \times a \text{ (multiply } n \text{ times)}$$

The number a is called the **base**, and n is called the **exponent** or the **power** of the expression. We can say that we raise a to the power n. In particular, $a^1 = a$ (we "repeat" number a one time). Also, $1^n = 1$ for any n. For two special cases, when power $n = 2$ and $n = 3$, we also say that a^2 is "a-squared", and a^3 is "a-cubed" respectively. The reason for that is a^2 represents the area of a square, and a^3 represents the volume of a cube with sides a.

The notation of exponents is useful in many situations, in particular, when we work with very big numbers (for example, with distances between planets). Below we show a way in which exponential expressions can also be used for very small numbers (such as, for example, distances inside molecules or atoms).

Let's consider examples and study some properties of exponents.

Example 3.1. Some people believe that one kilobyte (KB) of computer memory is equal to 1000 bytes (B). However, 1 KB = 1024 B, not 1000 B. The reason is that the number 1024 is a power of 2 but the number 1000 is not. Express number 1024 in exponential form with the base of 2.

Solution. Let's construct a table of exponential expressions with the base of 2 starting with the power of 5:

Power	5	6	7	8	9	10
Expression	$2^5 = 32$	$2^6 = 64$	$2^7 = 128$	$2^8 = 256$	$2^9 = 512$	$2^{10} = 1024$

Therefore, $1024 = 2^{10}$.

Example 3.2. It is known that the distance from our Earth to the Sun is about 150,000,000 km (150 million kilometers). Represent this distance in a short form using exponents.

Solution. We can write this number in the form $150,000,000 = 1.5 \times 100,000,000$. The number 100,000,000 contains 8 zeros and can be written as $100,000,000 = 10^8$.

Therefore, $150,000,000 = 1.5 \times 10^8$.

Note. Representation of big numbers such as in the example 3.2 in exponential form with the base of 10 is widely used in science. This form is called scientific notation.

In general, we say that a positive number a is in **scientific notation**, if it is written as a product of two parts:
   1) A number between 1 and 10 (1 is included, but 10 is not).
   2) Power of 10.

Example 3.3. Consider three numbers: 15.3×10^8, 0.15×10^6, and 2.73×10^4. Are these numbers in scientific notation?

Solution. It may seem that all three numbers are in scientific notation. However, it is not true. The first number 15.3×10^8 is not in scientific notation, because its first part, the number 15.3, is greater than 10, so it does not satisfy condition 1). The second number 0.15×10^6 is also not in scientific notation, since its first part, the number 0.15, is less than 1, so again it does not satisfy condition 1). The third number 2.73×10^4 is in scientific notation: its first part is 2.73, which is between 1 and 10.

Now, how about very small numbers? Consider, for example, the diameter of DNA helix. It is known that this diameter is about 0.0000002 cm. Is it possible somehow to represent this number also in a short form using exponents? The answer is yes. We will solve this problem in example 3.5 below. To come up with the idea how to do this we need to learn more about exponents. Let's start with some basic properties.

Basic Properties of Exponents

We will not give proofs here since proofs are very simple and follow directly from the definition of exponents (if you wish you can try to prove yourself). We will just illustrate the properties with some examples.

Consider how we can combine the product of a^2 and a^3 into one expression: $a^2 \times a^3 = (a \cdot a)(a \cdot a \cdot a) = a^5$. As you can see, to combine, we add powers 2 and 3, but not multiply them. A similar property holds according to the following general rule:

Product Rule. For any number a, and any positive integers n and m,

$$a^n \times a^m = a^{n+m}.$$

Note that both exponential expressions in this formula have the same base a. This restriction is very important. If, for example, you need to multiply $3^4 \times 2^5$, there is no simple rule to represent the answer as a single exponential expression. Also notice how product rule works: to multiply expressions with the same base, we **add** powers. A possible mistake here is to multiply powers instead of adding them.

Another rule is how to raise exponential expressions into a power. Consider the expression $\left(a^3\right)^2 = \left(a^3\right) \cdot \left(a^3\right) = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) = a^6$. This time, contrary to product rule, we **multiply** powers 3 and 2. Here is the general rule:

Power Rule. For any number a, and any positive integers n and m,

$$\left(a^n\right)^m = a^{n \cdot m}.$$

Now let's consider an example of dividing the exponential expressions:

$$a^6 \div a^2 = \frac{a^6}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a \cdot a \cdot a \cdot a = a^4.$$

To **divide** exponential expressions with the same base, we **subtract** powers (we **do not** divide them). The general rule is this:

Quotient Rule. For any nonzero number a, and any positive integers n and m, such that $n > m$,

$$\frac{a^n}{a^m} = a^{n-m}.$$

Note that in the above formula, the power in the numerator is greater than that in the denominator. But what if we need to divide expressions when the power of numerator is less than the power of denominator: $n < m$? One possible way is just to reduce this fraction by dividing numerator and denominator by a^n. Consider the example

$$\frac{a^2}{a^6} = \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a \cdot a \cdot a} = \frac{1}{a^4}.$$

In general, if $n < m$, then

$$a^n \div a^m = \frac{a^n}{a^m} = \frac{1}{a^{m-n}}.$$

In the next example, we will see how to raise the product of two numbers to a power:

$$(ab)^3 = (ab)(ab)(ab) = (aaa)(bbb) = a^3 b^3.$$

As you can see, we raise to the power 3 both numbers a and b. Here is the general rule.

Power of Product Rule. For any two numbers a and b, and any positive integer n,

$$(ab)^n = a^n b^n.$$

In similar way, if we have the quotient of numbers a and b, then to raise it to a power, we raise to that power both a and b, as in the following example:

$$\left(\frac{a}{b}\right)^2 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aa}{bb} = \frac{a^2}{b^2}.$$

In general, the following rule is true.

Power of Quotient Rule. For any number a, any nonzero b, and any positive integer n,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Negative Integer Exponents

Let's reconsider the Quotient Rule when the power n of the numerator is less than the power m of the denominator: $\dfrac{a^n}{a^m} = \dfrac{1}{a^{m-n}}$, $n < m$. It would be a good idea to somehow write this in the same exponential form as in the case when $n > m$: $\dfrac{a^n}{a^m} = a^{n-m}$. In doing this, we come up to the expressions with negative powers! For example, we can write

$$\frac{a^2}{a^6} = \frac{1}{a^4} = a^{2-6} = a^{-4}, \text{ or } a^{-4} = \frac{1}{a^4}.$$

You might say that negative exponent does not make any sense. Indeed, by the initial definition of exponential expressions, its power tells us how many times the base should be multiplied by itself. How can we multiply anything "negative number of times"? Of course, we cannot. However, we can get out of this situation if we simply accept by definition that $a^{-4} = \dfrac{1}{a^4}$. In general case, the definition of a negative exponent is similar to the above example for a^{-4}.

Definition. For any nonzero number a, and for any positive integer n, we define a^{-n} as

$$a^{-n} = \frac{1}{a^n}.$$

So, expressions with negative exponents are **reciprocals** to expressions with positive exponents. In particular,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}.$$

Now, let's try to invent the definition of a^0 (exponent is zero).

We can use similar approach as for negative exponents, using Quotient Rule for $m = n$. We will have $\dfrac{a^n}{a^n} = a^{n-n} = a^0$. Because $\dfrac{a^n}{a^n} = 1$, we get $a^0 = 1$. We come up with the

Definition. For any nonzero number a, $a^0 = 1$.

Example 3.4. Calculate. Write your answer as a decimal.

a) 10^0, b) 10^{-1}, c) 10^{-2}, d) 10^{-n} (n is a natural number).

Solution. By definition, we have

a) $10^0 = 1$, b) $10^{-1} = \dfrac{1}{10} = 0.1$, c) $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100} = 0.01$,

d) $10^{-n} = \dfrac{1}{10^n} = 0.0...01$ ($n - 1$ zeros after the decimal point).

Example 3.5. The diameter of DNA helix is about 0.0000002 cm. Represent this number in exponential form.

Solution. $0.0000002 = 2 \times 0.0000001 = 2 \times 10^{-7}$.

This example shows that expressions with negative exponents are useful for the representation of small numbers in a compact form. The above representation, as for big numbers, is also called scientific notation.

In conclusion, consider several examples. It can be shown that all the above properties of expressions with positive exponents are also true for negative exponents. In all problems below, the question is to simplify given expression and write the answer using positive exponents only. We assume that all bases of exponential expression are any nonzero numbers.

Example 3.6. $\dfrac{a^n}{a^{-m}}$.

Solution. We can use the Quotient Rule: $\dfrac{a^n}{a^{-m}} = a^{n-(-m)} = a^{n+m}$.

Technically, we can get the same answer as if we would "physically" move a^{-m} from the denominator up to the numerator and change the negative sign of m to positive. Then we can use the Product Rule: $\dfrac{a^n}{a^{-m}} = a^n a^m = a^{n+m}$. We will use similar method in the following example.

Example 3.7. $\dfrac{a^n b^{-p}}{a^{-m} b^q}$.

Solution. We can get rid of negative exponent for a^{-m} in denominator, and for b^{-p} in numerator, by moving these expressions into the opposite part of the fraction: move a^{-m} up to the numerator and move b^{-p} down to the denominator. Then apply the Product Rule. We will get

$$\frac{a^n b^{-p}}{a^{-m} b^q} = \frac{a^m a^n}{b^p b^q} = \frac{a^{m+n}}{b^{p+q}}.$$

Example 3.8. $\left(ax^{-5} y^3\right)\left(bxy^{-1}\right)$.

Solution. We can use Product Rule to combine x^{-5} and x (note that x can be written as x^1), and combine y^3 and y^{-1}:

$$\left(ax^{-5} y^3\right)\left(bxy^{-1}\right) = abx^{-5+1} y^{3-1} = abx^{-4} y^2.$$

Now, to get rid of negative exponent x^{-4}, similar to Example 3.7, move x^{-4} down:

$$abx^{-4} y^2 = \frac{aby^2}{x^4}.$$

Example 3.9. $\left(\dfrac{45u^{-3} v^8}{18u^{-6} v^{-4}}\right)^{-2}$.

Solution. It is possible to simplify this expression in different ways. As a first step, let's get rid of negative power -2 by applying the definition of negative exponent: take reciprocal of the fraction inside parentheses:

$$\left(\frac{45u^{-3} v^8}{18u^{-6} v^{-4}}\right)^{-2} = \left(\frac{18u^{-6} v^{-4}}{45u^{-3} v^8}\right)^{2}.$$

Next, we simplify fraction inside parentheses by reducing coefficients 18 and 45 by 9, and moving both exponents u^{-6} and v^{-4} down (it is not needed to move u^{-3} up). Then we use Product Rule:

$$\left(\frac{18u^{-6} v^{-4}}{45u^{-3} v^8}\right)^{2} = \left(\frac{2}{5u^{-3} u^6 v^8 v^4}\right)^{2} = \left(\frac{2}{5u^{-3+6} v^{8+4}}\right)^{2} = \left(\frac{2}{5u^3 v^{12}}\right)^{2}.$$

Finally, we use Power of Quotient and Power Rules:

$$\left(\frac{2}{5u^3 v^{12}}\right)^{2} = \frac{2^2}{5^2 \left(u^3\right)^2 \left(v^{12}\right)^2} = \frac{4}{25u^{3\cdot 2} v^{12\cdot 2}} = \frac{4}{25u^6 v^{24}}.$$

Exercises 3

In exercises 3.1 –3.4, write the given numbers in scientific notation.

3.1. The Earth's circumference at the equator is approximately 25,000 mi.

3.2. Mount Everest (on the border of Nepal and China) is the highest place on Earth above sea level, at about 29,000 ft.

3.3. A tiny space inside a computer chip has been measured to be 0.000256 cm wide.

3.4. A tiny space inside a computer chip has been measured to be 0.000014 cm long.

In exercises 3.5 – 3.8, write the given numbers as ordinary numbers.

3.5. The length of a bacterium is about 4×10^{-5} in.

3.6. The weight of a flea is about 8.75×10^{-2} g.

3.7. The diameter of the Moon is 3.475×10^{6} m.

3.8. The speed of sound in dry air at is 1.236×10^{3} km/h.

In Exercises 3.9 and 3.10, the numbers are **not** written in scientific notation (why?). Write these numbers in scientific notation.

3.9. a) 34.7×10^{-4}

   b) 0.25×10^{3}

3.10 a) 43.8×10^{-6}

   b) 0.36×10^{5}

In Exercises 3.11 and 3.12, calculate without calculator and write the answer without using of exponents. If the answer is not an integer, write it down as a fraction, not as a decimal.

3.11. a) $2 \cdot 3^{0}$

   b) $(2 \cdot 3)^{0}$

   c) 4^{-2}

   d) -4^{-2}

   e) $(-4)^{-2}$

3.12. a) $3 \cdot 4^{0}$

   b) $(3 \cdot 4)^{0}$

   c) 5^{-2}

   d) -5^{-2}

   e) $(-5)^{-2}$

In Exercises 3.13 – 3.24, simplify and write the answer using **positive** exponents only (assume that all letters represent positive numbers).

3.13. a) $a^{5} a^{-7}$

   b) $c^{-3} c^{-6}$

3.14. a) $c^{-6} c^{3}$

   b) $n^{-2} n^{-4}$

c) $\left(a^3\right)^{-2}$

d) $\left(-\dfrac{m}{n}\right)^{-2}$

c) $\left(d^{-4}\right)^3$

d) $\left(-\dfrac{a}{b}\right)^{-3}$

3.15. a) $\dfrac{p^8}{p^4}$

b) $\dfrac{p^{-8}}{p^4}$

c) $\dfrac{p^8}{p^{-4}}$

d) $\dfrac{p^{-8}}{p^{-4}}$

e) $\dfrac{p^4}{p^8}$

f) $\dfrac{p^4}{p^{-8}}$

g) $\dfrac{p^{-4}}{p^8}$

h) $\dfrac{p^{-4}}{p^{-8}}$

3.16. a) $\dfrac{r^9}{r^3}$

b) $\dfrac{r^9}{r^{-3}}$

c) $\dfrac{r^{-9}}{r^3}$

d) $\dfrac{r^{-9}}{r^{-3}}$

e) $\dfrac{r^3}{r^9}$

f) $\dfrac{r^{-3}}{r^9}$

g) $\dfrac{r^3}{r^{-9}}$

h) $\dfrac{r^{-3}}{r^{-9}}$

3.17. $\dfrac{x^{-a}y^b}{x^c y^{-d}}$

3.18. $\dfrac{p^w q^{-x}}{p^{-y}q^z}$

3.19 $\left(ma^4 b^{-2}\right)\left(na^{-5}b^3\right)$

3.20. $\left(xu^{-5}v^3\right)\left(yu^7 v^{-6}\right)$

3.21. $\left(-\dfrac{35r^6 s^{-12}}{42r^{-2}s^3}\right)^{-2}$

3.22. $\left(-\dfrac{12x^{-15}y^9}{30x^5 y^{-3}}\right)^{-3}$

3.23. $\left(-\dfrac{24p^{-8}q^{14}}{16p^{-2}q^7}\right)^{-3}$

3.24. $\left(-\dfrac{25m^{-12}n^8}{15m^{-2}n^4}\right)^{-2}$

Challenge Problem

3.25. Solve the equation $(x^2 - 7x + 11)^{x^2 - 13x + 42} = 1$.

Hint: There are six roots. To find them, consider three cases.

Session 4

Rational Expressions and Complex Fractions

Recall that a **rational number** is a number that can be written as a fraction of integers (having a **numerator** on the top and a **denominator** on the bottom). Usually, we write a fraction in the form $\dfrac{m}{n}$, where m and n are two integers (m is the numerator, and n is the denominator and $n \neq 0$). We treat a fraction as a ratio of its numerator to denominator, so, we can write $\dfrac{m}{n} = m \div n$. We will always assume that the denominator n is not equal to zero.

We consider here **rational expressions**. These are also fractions. However, their numerators and denominators are not necessarily numbers. They are expressions that are called **polynomials**. A polynomial is an expression that can be written in a form that contains a variable, say x, together with the operations of addition, subtraction, and multiplication of x by numbers and by itself. When x is multiplied by itself, (like $x \cdot x$, $x \cdot x \cdot x$,...), we usually write this as an exponential expression (like x^2, x^3,...). Here are some examples of polynomials:

$$\frac{1}{2}x - 3, \ 5x^3 - 3x + 2, \ 3x^2 - 7x + 4.$$

The last polynomial is called a **quadratic trinomial** (quadratic, because the highest power of x is 2, and trinomial, since it contains three terms). The expression $x(3x^2 + x)$ is also a polynomial, because after distribution it can be written as $3x^3 + x^2$. On the contrary, $3 + \dfrac{1}{x}$ is not a polynomial since it contains the variable x in the denominator (so, division by the variable) and cannot be reduced to a polynomial.

Definition. A rational expression is a ratio of two polynomials (or a fraction whose numerator and denominator are polynomials). Here are several examples of rational expressions:

$$\frac{5x^2 - 3x + 2}{2x - 1}, \ \frac{1}{x}, \ -\frac{3}{x^2 - 1}, \ \frac{3x + 2}{x^3 - 5x^2}.$$

Below we consider examples on how to add and subtract rational expressions. Mostly, these operations can be performed in a manner similar to rational numbers (ordinary fractions). Where possible, we will point out the similarity between rational numbers and rational expressions.

Simplification of Rational Expressions

When adding or subtracting, we will also simplify (if possible) resulting expressions. In order to simplify, we factor the numerator and denominator, and reduce (cancel out) a common factor, as we were doing with rational numbers (ordinary fractions). Let's see some examples of simplification.

Example 4.1. Simplify $\dfrac{10x^2 - 15x}{20x}$.

Solution. A possible mistake here is to cancel x from $15x$ (in the numerator) and $20x$ (in the denominator), resulting in a wrong answer $(10x^2 - 15)/20$. As we mentioned above, the possible correct way is to factor the numerator before reducing. We can factor out $5x$ from the numerator and then reduce:

$$\frac{10x^2 - 15x}{20x} = \frac{5x(2x-3)}{20x} = \frac{5x(2x-3)}{5x \cdot 4} = \frac{2x-3}{4}.$$

Example 4.2. Simplify $\dfrac{16x^2 + 12x}{6x - 10}$.

Solution. Here we can factor out both the numerator and denominator and then reduce (divide the numerator and denominator) by 2:

$$\frac{16x^2 + 12x}{6x - 10} = \frac{4x(4x+3)}{2(3x-5)} = \frac{2x(4x+3)}{3x-5}.$$

Example 4.3. Simplify $\dfrac{5x^2 + 30x + 40}{3x^2 - 3x - 18}$.

Solution. As in the previous problem, we start with the factoring of the numerator and the denominator. We can factor them in two steps. First, factor out 5 from the numerator and 3 from the denominator:

$$\frac{5x^2 + 30x + 40}{3x^2 - 3x - 18} = \frac{5(x^2 + 6x + 8)}{3(x^2 - x - 6)}.$$

The second step is to factor the quadratic trinomials inside the parentheses:

$$x^2 + 6x + 8 = (x+2)(x+4) \text{ and } x^2 - x - 6 = (x+2)(x-3).$$

We can complete the factorization and then reduce (cancel) the common factor $x + 2$:

$$\frac{5(x^2 + 6x + 8)}{3(x^2 - x - 6)} = \frac{5(x+2)(x+4)}{3(x+2)(x-3)} = \frac{5(x+4)}{3(x-3)}.$$

Addition and Subtraction of Rational Expressions

Example 4.4. Add $\quad \dfrac{5x}{2x-1} + \dfrac{3}{2x-1}$.

Solution. Recall that it is very easy to add (or subtract) numerical fractions if they have the same denominator: just add (or subtract) numerators and keep (do not add or subtract) their common denominator. For example, $\dfrac{2}{7} + \dfrac{3}{7} = \dfrac{5}{7}$, $\dfrac{5}{9} - \dfrac{7}{9} = -\dfrac{2}{9}$.

The same rule applies to rational expressions. For given example,

$$\frac{5x}{2x-1} + \frac{3}{2x-1} = \frac{5x+3}{2x-1}.$$

Example 4.5. Subtract $\quad \dfrac{3a+4}{8} - \dfrac{a-2}{6}$.

Solution. This time the denominators are different. Recall how we would subtract rational numbers (fractions), let's say $\dfrac{7}{8} - \dfrac{5}{6}$.

To subtract, we replace these fractions with equivalent ones having the same denominator, which is called the **LCD** (Least Common Denominator). LCD is the smallest number that is divisible by both denominators. Technically, we can subtract fractions in three steps.

1) Find the LCD. For given denominators 8 and 6, the LCD = 24. We put the LCD into the denominator of the resulting fraction.

2) Find the **complements** of each denominator to the LCD. A complement is the number such that if we multiply it by the denominator, we get the LCD. To find complements, just divide the LCD by each denominator. For the denominator 8, the complement is 3 ($24 \div 8 = 3$), and for the denominator 6, the complement is 4 ($24 \div 6 = 4$).

3) Calculate the numerator of the resulting fraction: multiply numerator of each fraction by the complement to its denominator and subtract the results. For the given fractions 7/8 and 5/6, multiply the numerator 7 by 3 (complement to the denominator 8), and the numerator 5 by 4 (complement to the denominator 6):

$$\frac{7}{8} - \frac{5}{6} = \frac{7 \cdot 3 - 5 \cdot 4}{24} = \frac{21 - 20}{24} = \frac{1}{24}.$$

To subtract rational expressions, we do the same thing:

$$\frac{3a+4}{8} - \frac{a-2}{6} = \frac{(3a+4)3 - (a-2)4}{24} = \frac{9a+12-4a+8}{24} = \frac{5a+20}{24}.$$

Now consider an example when the denominators are different and contain variables.

Example 4.6. Add $\dfrac{3}{10x} + \dfrac{4}{15y}$.

Solution. As before, we first construct the LCD. The denominators contain both numbers and letters. Therefore, the LCD consists of two parts: numerical part and letter part. For the numbers 10 and 15, the numerical part of the LCD is 30. The letters x and y do not have common factors. Therefore, the letter part of the LCD is their product xy. The entire LCD is the product of the numerical and letter parts:

$$LCD = 30xy.$$

Next, we find complements of each denominator to the LCD by dividing the LCD by denominators.

For the denominator $10x$, the complement is $30xy/10x = 3y$.

For the denominator $15y$, the complement is $30xy/15y = 2x$.

 Finally, we add the given fractions:

$$\frac{3}{10x} + \frac{4}{15y} = \frac{3 \cdot 3y + 4 \cdot 2x}{30xy} = \frac{9y + 8x}{30xy}.$$

Example 4.7. Combine $\dfrac{5}{a^2} - \dfrac{11}{6a} + \dfrac{9}{14}$.

Solution. To construct the LCD, similarly to the previous example, we construct separately its numerical and letter parts.

For the numbers 6 and 14 in denominators, the numerical part of the LCD is 42.

For the letters a^2 and a, the letter part of the LCD is a^2 .

The entire LCD is the product of both parts: LCD = $42a^2$.

Next, we find complements for each denominator to the LCD:

For a^2 , the complement is $42a^2 / a^2 = 42$.

For $6a$, the complement is $42a^2 / 6a = 7a$.

For 14, the complement is $42a^2 / 14 = 3a^2$.

From here,

$$\frac{5}{a^2} - \frac{11}{6a} + \frac{9}{14} = \frac{5 \cdot 42 - 11 \cdot 7a + 9 \cdot 3a^2}{42a^2} = \frac{210 - 77a + 27a^2}{42a^2}.$$

Note. In general, if the denominators contain exponential expressions with the same base and different powers, include into the LCD the expression with the **biggest** power. Thus, in the previous example, for the expressions a^2 and a in the denominators we have included a^2 into the LCD.

If denominators do not contain common factors, include into the LCD the product of both denominators.

Example 4.8. Subtract $\dfrac{4}{5x-3} - \dfrac{2}{3x-5}$.

Solution. The denominators $5x-3$ and $3x-5$ do not have common factors, therefore, the LCD is simply their product:

$$\text{LCD} = (5x-3)(3x-5).$$

The denominators $5x-3$ and $3x-5$ are complemented to each other, therefore

$$\frac{4}{5x-3} - \frac{2}{3x-5} = \frac{4(3x-5)-2(5x-3)}{(5x-3)(3x-5)} = \frac{12x-20-10x+6}{(5x-3)(3x-5)} = \frac{2x-14}{(5x-3)(3x-5)}.$$

Example 4.9. Subtract $\dfrac{y-5}{y-6} - \dfrac{y+5}{6-y}$.

Solution. Notice that the denominators of these fractions are "almost the same" (they differ only in signs and the order of terms). We can make them exactly the same by using the following connection between expressions $a-b$ and $b-a$:

$$a-b = -(b-a).$$

Thus, $6-y = -(y-6)$ and

$$\frac{y-5}{y-6} - \frac{y+5}{6-y} = \frac{y-5}{y-6} - \frac{y+5}{-(y-6)} = \frac{y-5}{y-6} + \frac{y+5}{y-6} = \frac{2y}{y-6}.$$

Example 4.10. Add $\dfrac{7}{x-3} + 4$.

Solution. We can treat the integer 4 as a fraction with the denominator 1: $4 = \dfrac{4}{1}$.

From here, the LCD of the denominators $x-3$ and 1 is $x-3$, and these denominators are complemented to each other. Therefore,

$$\frac{7}{x-3} + 4 = \frac{7}{x-3} + \frac{4}{1} = \frac{7 \cdot 1 + 4(x-3)}{x-3} = \frac{7+4x-12}{x-3} = \frac{4x-5}{x-3}.$$

Example 4.11. Add $\dfrac{5x}{x^2-4} + \dfrac{3}{2x-4}$.

Solution. At the first glance, it might seem that the denominators x^2-4 and $2x-4$ do not have common factors. However, they do! To see that, factor both. It is easy to factor the denominator $2x-4$ by factoring the number 2: $2x-4 = 2(x-2)$. The denominator x^2-4 can be factored using the formula for the difference of two squares:

$$a^2 - b^2 = (a-b)(a+b).$$

From here, $x^2 - 4 = (x-2)(x+2)$. Now, compare the denominators in the factored form:

$$2(x-2) \text{ and } (x-2)(x+2).$$

We see the common factor $x-2$. This is a part of the LCD. Also, we put the other factors 2 and $x+2$ into the LCD. Therefore, the entire LCD $= 2(x-2)(x+2)$. Next, we find the complements for each denominator:

For $x^2 - 4 = (x-2)(x+2)$, the complement is 2.

For $2x - 4 = 2(x-2)$, the complement is $x+2$.

Finally, we add the fractions:

$$\frac{5x}{x^2-4} + \frac{3}{2x-4} = \frac{5x \cdot 2 + 3(x+2)}{2(x-2)(x+2)} = \frac{10x + 3x + 6}{2(x-2)(x+2)} = \frac{13x+6}{2(x-2)(x+2)}.$$

Example 4.12. Subtract $\dfrac{7}{x^2-x-12} - \dfrac{5}{x^2+x-6}$.

Solution. Again, as a first step, we factor each denominator:

$$x^2 - x - 12 = (x+3)(x-4) \text{ and } x^2 + x - 6 = (x+3)(x-2).$$

Now, to construct the LCD, we multiply all factors from both denominators (taking the common factor $x+3$ only one time):

$$\text{LCD} = (x+3)(x-4)(x-2).$$

Next, using the LCD, we find the complements of each denominator.

For $x^2 - x - 12 = (x+3)(x-4)$, the complement is $x - 2$.

For $x^2 + x - 6 = (x+3)(x-2)$, the complement is $x - 4$.

Finally, we subtract the given fractions:

$$\frac{7}{x^2-x-12} - \frac{5}{x^2+x-6} = \frac{7(x-2) - 5(x-4)}{(x+3)(x-4)(x-2)}$$

$$= \frac{7x - 14 - 5x + 20}{(x+3)(x-4)(x-2)} = \frac{2x+6}{(x+3)(x-4)(x-2)}.$$

We can simplify the last expression even more by factoring the numerator $2x + 6$ as $2(x+3)$ and then canceling the factor $x + 3$ from the numerator and the denominator:

$$\frac{2x+6}{(x+3)(x-4)(x-2)} = \frac{2(x+3)}{(x+3)(x-4)(x-2)} = \frac{2}{(x-4)(x-2)}.$$

Example 4.13. Combine $\dfrac{3}{a^2-4a-12}+\dfrac{4}{a-6}-\dfrac{2}{a+2}$.

Solution. Factor the first denominator:

$$a^2-4a-12=(a-6)(a+2) .$$

The factors $a-6$ and $a+2$ are the denominators of the second and the third fractions, so

$$\text{LCD}=(a-6)(a+2).$$

Since the LCD coincides with the first denominator, the complement of this is 1, and we need to find complements only for the second $(a-6)$ and the third $(a+2)$ denominators.

For $a-6$, the complement is $a+2$.

For $a+2$, the complement is $a-6$.

From here

$$\frac{3}{a^2-4a-12}+\frac{4}{a-6}-\frac{2}{a+2}=\frac{3+4(a+2)-2(a-6)}{(a-6)(a+2)}$$

$$=\frac{3+4a+8-2a+12}{(a-6)(a+2)}=\frac{2a+23}{(a-6)(a+2)} .$$

Complex Fractions

Complex fractions are fractions that contain other fractions in their numerators and/or denominators. We will call fractions inside a complex fraction **inner ones**. For example,

fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are inner fractions in the complex fraction: $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$ We will consider

methods on how to simplify complex fractions.

Example 4.14. Simplify $\dfrac{\dfrac{7}{15ab^2}}{\dfrac{14}{27ab}}$.

Solution. We can represent this complex fraction in the form of division of its numerator by its denominator: $\dfrac{7}{15ab^2}\div\dfrac{14}{27ab}$. Now we use the rule for division of fractions: multiply the first fraction by the reciprocal of the second: $\dfrac{7}{15ab^2}\cdot\dfrac{27ab}{14}$. After simplification (reducing) we get the final answer $\dfrac{9}{10b}$.

Note. Working with complex fractions, it is important not to confuse those that look similar, but are actually different. We need to carefully identify the position of the "main" (longest) fraction line. This line shows the place where we divide numerator by denominator. For example, compare the following complex fractions (which have different "longest" lines):

$$\frac{\dfrac{a}{b}}{c} = \frac{a}{b} \div c = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}. \qquad\qquad \frac{a}{\dfrac{b}{c}} = a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{ac}{b}.$$

As you see, these fractions are different.

In the examples below, to simplify complex fractions, we use two methods.

First method: find the LCD for all inner fractions contained in the numerator and the denominator of the complex fraction, and then multiply the numerator and the denominator of the complex fraction by this LCD.

Second method: simplify the numerator and the denominator of a complex fraction separately, and then divide its numerator by the denominator.

Example 4.15. Simplify $\dfrac{\dfrac{2}{3} - \dfrac{5}{6}}{\dfrac{4}{15} + \dfrac{7}{12}}$.

Solution.

First method. The denominators of the four inner fractions are 3, 6, 15, and 12. Their LCD is 60. Multiply the numerator and denominator of the complex fraction by this LCD:

$$\frac{60\left(\dfrac{2}{3} - \dfrac{5}{6}\right)}{60\left(\dfrac{4}{15} + \dfrac{7}{12}\right)} = \frac{60 \cdot \dfrac{2}{3} - 60 \cdot \dfrac{5}{6}}{60 \cdot \dfrac{4}{15} + 60 \cdot \dfrac{7}{12}} = \frac{40 - 50}{16 + 35} = \frac{-10}{51} = -\frac{10}{51}.$$

Second method. Simplify the numerator and denominator separately:

$$\frac{2}{3} - \frac{5}{6} = \frac{2 \cdot 2 - 5}{6} = -\frac{1}{6}, \qquad \frac{4}{15} + \frac{7}{12} = \frac{4 \cdot 4 + 7 \cdot 5}{60} = \frac{51}{60}.$$

Then divide numerator by denominator: $-\dfrac{1}{6} \div \dfrac{51}{60} = -\dfrac{1}{6} \cdot \dfrac{60}{51} = -\dfrac{10}{51}.$

Example 4.16. Simplify $\dfrac{\dfrac{3}{x^2 y} + \dfrac{2}{xy}}{\dfrac{5}{xy^2} - \dfrac{4}{x^2}}$.

Solution.

First method. The denominators of the four inner fractions are x^2y, xy, xy^2, and x^2. Their LCD is x^2y^2. Multiply the numerator and the denominator of the above complex fraction by this LCD:

$$\frac{x^2y^2\left(\dfrac{3}{x^2y}+\dfrac{2}{xy}\right)}{x^2y^2\left(\dfrac{5}{xy^2}-\dfrac{4}{x^2}\right)}=\frac{\dfrac{3x^2y^2}{x^2y}+\dfrac{2x^2y^2}{xy}}{\dfrac{5x^2y^2}{xy^2}-\dfrac{4x^2y^2}{x^2}}=\frac{3y+2xy}{5x-4y^2}.$$

Second method.

Add inner fractions in the numerator: $\dfrac{3}{x^2y}+\dfrac{2}{xy}=\dfrac{3+2x}{x^2y}$.

Subtract inner fractions in the denominator: $\dfrac{5}{xy^2}-\dfrac{4}{x^2}=\dfrac{5x-4y^2}{x^2y^2}$.

Divide numerator by denominator:

$$\frac{3+2x}{x^2y}\div\frac{5x-4y^2}{x^2y^2}=\frac{3+2x}{x^2y}\cdot\frac{x^2y^2}{5x-4y^2}=\frac{(3+2x)y}{5x-4y^2}=\frac{3y+2xy}{5x-4y^2}.$$

Example 4.17. Simplify $\dfrac{6-\dfrac{5}{n-2}}{7+\dfrac{3}{n-2}}$.

Solution.

First method. The LCD of the inner denominators is $n-2$. Multiply the numerator and denominator of the complex fraction by $n-2$:

$$\frac{6-\dfrac{5}{n-2}}{7+\dfrac{3}{n-2}}=\frac{(n-2)\left(6-\dfrac{5}{n-2}\right)}{(n-2)\left(7+\dfrac{3}{n-2}\right)}=\frac{6(n-2)-5}{7(n-2)+3}=\frac{6n-12-5}{7n-14+3}=\frac{6n-17}{7n-11}.$$

Second method.

Combine the numerator: $6-\dfrac{5}{n-2}=\dfrac{6(n-2)-5}{n-2}=\dfrac{6n-17}{n-2}$.

Combine the denominator:

$$7 + \frac{3}{n-2} = \frac{7(n-2)+3}{n-2} = \frac{7n-14+3}{n-2} = \frac{7n-11}{n-2}.$$

Divide numerator by denominator:

$$\frac{6n-17}{n-2} \div \frac{7n-11}{n-2} = \frac{6n-17}{n-2} \cdot \frac{n-2}{7n-11} = \frac{6n-17}{7n-11}.$$

Exercises 4

In exercises 4.1 – 4.8, simplify the given expressions.

4.1. $\dfrac{18y^2 - 12y}{15y}$

4.2. $\dfrac{24z^2 - 16z}{14z}$

4.3. $\dfrac{15z^2 + 25z}{10z - 30}$

4.4. $\dfrac{12y^2 + 18y}{9y - 15}$

4.5. $\dfrac{4x^2 + 8x - 60}{6x^2 + 6x - 72}$

4.6. $\dfrac{6x^2 + 24x - 72}{8x^2 + 40x - 48}$

4.7. $\dfrac{2x^2 - 6x - 56}{7x^2 - 49x}$

4.8. $\dfrac{3x^2 - 3x - 36}{4x^2 - 16x}$

4.9. Add and simplify

$\dfrac{6x}{3x + 2} + \dfrac{4}{3x + 2}$

4.10. Add and simplify

$\dfrac{10x}{5x + 1} + \dfrac{2}{5x + 1}$

4.11. Subtract

$\dfrac{4m + 3}{12} - \dfrac{m - 3}{18}$

4.12. Subtract

$\dfrac{2n + 5}{10} - \dfrac{n - 4}{12}$

4.13. Add

$\dfrac{5}{8x} + \dfrac{7}{12y}$

4.14. Add

$\dfrac{7}{6x} + \dfrac{3}{8y}$

4.15. Combine

$\dfrac{4}{15x^2} - \dfrac{5}{3x} + \dfrac{7}{6}$

4.16. Combine

$\dfrac{5}{12x^2} - \dfrac{7}{6x} + \dfrac{3}{8}$

4.17. Subtract

$\dfrac{3}{4x - 7} - \dfrac{5}{7x - 4}$

4.18. Subtract

$\dfrac{6}{3x - 4} - \dfrac{7}{4x - 3}$

4.19. Subtract

$\dfrac{a - 2}{a - 4} - \dfrac{a + 2}{4 - a}$

4.20. Subtract

$\dfrac{b - 4}{b - 5} - \dfrac{b + 4}{5 - b}$

4.21. Add $\quad \dfrac{6}{x - 2} + 3$

4.22. Add $\quad \dfrac{8}{x - 4} + 2$

4.23. Add and simplify

$$\frac{x-2}{5x+25}+\frac{3x+1}{x^2-25}$$

4.24. Add and simplify

$$\frac{4x+2}{x^2-16}+\frac{x-3}{4x+16}$$

4.25. Subtract and simplify

$$\frac{3}{x^2+7x+10}-\frac{1}{x^2+5x+6}$$

4.26. Subtract and simplify

$$\frac{6}{x^2+4x-5}-\frac{5}{x^2+3x-4}$$

4.27. Combine and simplify

$$\frac{5}{b^2+2b-8}+\frac{3}{b+4}-\frac{3}{b-2}$$

4.28. Combine and simplify

$$\frac{7}{c^2+2c-15}-\frac{4}{c-3}+\frac{4}{c+5}$$

In exercises 4.29 – 4.48, simplify the complex fractions.

4.29. $\dfrac{\dfrac{5}{12c^3d}}{\dfrac{15}{16cd^2}}$

4.30. $\dfrac{\dfrac{18}{5m^6n}}{\dfrac{6}{25m^3n^2}}$

4.31. $\dfrac{\dfrac{3}{x}+\dfrac{2}{x^2}}{\dfrac{4}{x^2}-\dfrac{5}{x}}$

4.32. $\dfrac{\dfrac{2}{x}-\dfrac{3}{x^2}}{\dfrac{5}{x^2}+\dfrac{4}{x}}$

4.33. $\dfrac{\dfrac{7x}{x+5}}{\dfrac{3}{2x+10}}$

4.34. $\dfrac{\dfrac{5x}{x-7}}{\dfrac{4}{3x-21}}$

4.35. $\dfrac{\dfrac{2}{5}-\dfrac{3}{4}}{\dfrac{7}{10}+\dfrac{1}{2}}$

4.36. $\dfrac{\dfrac{5}{6}+\dfrac{7}{8}}{\dfrac{3}{4}-\dfrac{2}{3}}$

4.37. $\dfrac{\dfrac{4}{xy^2}+\dfrac{3}{xy}}{\dfrac{6}{x^2y}-\dfrac{2}{y^2}}$

4.38. $\dfrac{\dfrac{6}{xy}-\dfrac{5}{x^2y}}{\dfrac{4}{x^2}+\dfrac{3}{xy^2}}$

4.39. $\dfrac{6-\dfrac{5}{2x}}{\dfrac{5}{6x}-2}$

4.40. $\dfrac{12-\dfrac{7}{2y}}{\dfrac{7}{8y}-3}$

4.41. $\dfrac{4 - \dfrac{7}{k-3}}{6 + \dfrac{5}{k-3}}$

4.42. $\dfrac{5 + \dfrac{8}{m-4}}{7 - \dfrac{6}{m-4}}$

4.43. $\dfrac{1 + \dfrac{1}{x} - \dfrac{2}{x^2}}{1 + \dfrac{6}{x} + \dfrac{8}{x^2}}$

4.44. $\dfrac{1 + \dfrac{2}{x} - \dfrac{3}{x^2}}{1 + \dfrac{5}{x} + \dfrac{6}{x^2}}$

4. 45. $\dfrac{\dfrac{2}{x} + \dfrac{3}{x+4}}{\dfrac{7}{x+4} - \dfrac{2}{x}}$

4.46. $\dfrac{\dfrac{3}{x} + \dfrac{4}{x+2}}{\dfrac{9}{x+2} - \dfrac{2}{x}}$

4. 47. $\dfrac{\dfrac{a}{x} + \dfrac{b}{x+c}}{\dfrac{d}{x+c} - \dfrac{e}{x}}$

4.48. $\dfrac{\dfrac{a}{x-b} - \dfrac{c}{x}}{\dfrac{d}{x} + \dfrac{e}{x-b}}$

Challenge Problems

4.49. Let $\dfrac{A}{B} = \dfrac{a}{b}$. Prove that

$$\frac{A}{x^2 - ax} - \frac{B}{x^2 - bx} = \frac{A - B}{(x-a)(x-b)}.$$

4.50. Let $d = bc - 2a - 2b$. Prove that

$$\frac{x - a}{bx + b^2} + \frac{cx + d}{x^2 - b^2} = \frac{x + a + d}{b(x - b)}.$$

4.51. Let $b = cd - ce - ab$. Prove that

$$\frac{ax + b}{cx - cd} - \frac{x - e}{x - d} = \frac{a - c}{c}.$$

4.52. Let $d = b(2a - 2b + c)$. Prove that

$$\frac{x + a}{x + b} + \frac{cx + d}{x^2 - b^2} = \frac{x + a + c - 2b}{x - b}.$$

4.53. Let a, b, and c be natural numbers such that c is not equal to 1. Compare the values

$$\frac{\dfrac{a}{b}}{c} \qquad \text{and} \qquad \dfrac{a}{\dfrac{b}{c}}$$

Are they equal? If not, which one is bigger?

Session 5

Rational Equations

In the previous session, we worked with rational **expressions**. In this session we will work with **rational (fractional) equations**. Both sides of such equations are sums or differences of rational expressions. Here is an example: $\dfrac{x+5}{4} = \dfrac{3}{4} - \dfrac{x-7}{6}$. Below we solve this equation by reducing it to an equation with no fractions. As with expressions, we will use the LCD to do this. However, the main technical difference here is that we must keep the LCD in the expression (and write it in the denominator of the answer), while we can drop the LCD in an equation.

The reason why we can drop the denominator is this. When keeping the LCD (as we do with expressions), both sides of the equation become fractions with the same denominator (which is the LCD). If two fractions are equal and have the same denominator, then their numerators are also equal, so we equate numerators and drop denominators. Here is a simple example: $\dfrac{x}{4} = \dfrac{5}{4}$. Form here we conclude that $x = 5$. So, technically, we drop the common denominator 4 and equate the numerators.

Example 5.1. Solve the equation $\dfrac{x+5}{4} = \dfrac{3}{4} - \dfrac{x-7}{6}$.

Solution. The first step is the same as for expressions: find the LCD. For the denominators 4 and 6, the LCD = 12 (notice that the number 24 is also common denominator but not the least). The second step is also the same: find complements for each denominator to the LCD.

For the denominator 4, the complement is 3,

For the denominator 6, the complement is 2.

The third step is again the same: multiply each numerator by the corresponding complement. But now, unlike what we did with expressions, **we may drop all denominators!** As a result, the original equation becomes an equation with no fractions:

$$3(x+5) = 3\cdot 3 - 2(x-7).$$

Now it is easy to solve it:

$$3x + 3\cdot 5 = 3\cdot 3 - 2x + 2\cdot 7 \implies 3x + 15 = 9 - 2x + 14 \implies 3x + 15 = 23 - 2x.$$

From this point we collect all terms with the variable x on the left side, and all other terms (numbers) on the right side by using the moving method:

$$3x + 2x = 23 - 15 \implies 5x = 8 \implies x = \dfrac{8}{5}.$$

Next, we consider equations in which denominators contain a variable. The technique here is the same. The only additional (and very important) thing is that we need to check that

the final answer does not cause any denominator of the original equation to become zero. If this happens, we must reject such a solution.

Example 5.2. Solve the equation $\dfrac{2}{7x}+\dfrac{1}{6}=\dfrac{11}{14x}$.

Solution. For the denominators $7x$, 6 and $14x$, the LCD $= 42x$. Next, we find complements for each denominator to the LCD:

For $7x$, the complement is 6.

For 6, the complement is $7x$.

For $14x$, the complement is 3.

We multiply each numerator by its corresponding complement and drop the LCD. The equation becomes free of fractions:

$$2\cdot 6+1\cdot 7x=11\cdot 3$$

Now we solve this equation:

$$12+7x=33 \;\Rightarrow\; 7x=33-12 \;\Rightarrow\; 7x=21 \;\Rightarrow\; x=3 .$$

None of the denominators of the original equation is zero for this value of x, so $x = 3$ is the final answer.

Example 5.3. Solve the equation $\dfrac{3}{4}+\dfrac{5}{x-5}=\dfrac{x}{x-5}$.

Solution. For the denominators 4 and $x - 5$, the LCD $= 4(x - 5)$.

Complements for denominators to the LCD are:

For 4, the complement is $x - 5$.

For $x - 5$, the complement is 4.

The equation becomes

$$3(x-5)+5\cdot 4=4x .$$

We solve it:

$$3x-15+20=4x \;\Rightarrow\; 3x+5=4x \;\Rightarrow\; 3x-4x=-5 ,$$

$$-x=-5 \;\Rightarrow\; x=5 .$$

For $x = 5$, the denominator $x - 5$ becomes zero, so we reject the value $x = 5$. Because this is the only possible solution, the original equation has no solution at all.

Example 5.4. Solve the equation $\dfrac{3}{2}+\dfrac{4}{x^2-16}=\dfrac{3x}{2x-8}$.

Solution.

1) To find the LCD, factor the second and third denominators.
   Second denominator:

$$x^2-16=(x-4)(x+4) .$$

   Third denominator:

$$2x - 8 = 2(x - 4).$$

The LCD of all three denominators is $2(x-4)(x+4)$.

2) Find the complement for each denominator to LCD:

For 2, the complement is $(x-4)(x+4) = x^2 - 16$.

For $x^2 - 16 = (x-4)(x+4)$, the complement is 2.

For $2x - 8 = 2(x-4)$, the complement is $x+4$.

3) Multiply each numerator of the original equation by the corresponding complement and drop the denominator. The equation becomes

$$3(x^2 - 16) + 4 \cdot 2 = 3x(x+4).$$

4) Solve the above equation:

$$3x^2 - 3 \cdot 16 + 4 \cdot 2 = 3x^2 + 3x \cdot 4 \implies 3x^2 - 48 + 8 = 3x^2 + 12x,$$

$$3x^2 - 3x^2 - 12x = 48 - 8 \implies -12x = 40 \implies x = -\frac{40}{12} = -\frac{10}{3}.$$

5) Looking at the denominators of the original equation, you may see that none of the denominators is zero if $x = -\dfrac{10}{3}$ and this is the solution.

Example 5.5. Solve the equation $\dfrac{x}{x-3} + \dfrac{1}{x+2} = \dfrac{18-x}{x^2 - x - 6}$.

Solution.

1) To find the LCD, factor the third denominator: $x^2 - x - 6 = (x-3)(x+2)$. You may notice that the first factor $x - 3$ is the denominator of the first fraction, and the second factor $x + 2$ is the denominator of the second fraction, so
$$\text{LCD} = x^2 - x - 6 = (x-3)(x+2).$$

2) Find the complement for each denominator to the LCD:

For $x - 3$, the complement is $x + 2$.

For $x + 2$, the complement is $x - 3$.

For $x^2 - x - 6$ the complement is 1.

3) Multiply each numerator of the original equation by the corresponding complement and drop the denominator. The equation becomes

$$x(x+2) + 1 \cdot (x-3) = 18 - x.$$

4) Solve the above equation:

$$x^2 + 2x + x - 3 = 18 - x \implies x^2 + 3x - 3 = 18 - x,$$

$$x^2 + 3x - 3 - 18 + x = 0 \implies x^2 + 4x - 21 = 0.$$

The last equation is a quadratic equation. We can solve it by factoring:

$$(x+7)(x-3) = 0 \implies x = -7 \text{ and } x = 3.$$

5) Check whether these numbers make any of the denominators of the original equation zero. The number -7 does not, but 3 does, so we must reject the number 3.
Final answer: the original equation has only one solution $x = -7$.

Example 5.6. Solve the equation $\dfrac{3}{4n+20} - \dfrac{2}{n-3} = \dfrac{6}{n^2+2n-15}$.

Solution.

1) To find LCD, first we factor denominators, which are factorable. It is possible to factor the first and third denominators:

$$4n+20 = 4(n+5) \text{ and } n^2+2n-15 = (n+5)(n-3).$$

2) Now we construct the LCD by multiplying all of the above factors (taking the common factor $n + 5$ only one time):

$$\text{LCD} = 4(n+5)(n-3).$$

3) Find the complement for each denominator to the LCD:

For $4n+20 = 4(n+5)$, the complement is $n - 3$.

For $n - 3$, the complement is $4(n+5) = 4n+20$.

For $n^2+2n-15 = (n+5)(n-3)$, the complement is 4.

4) Multiply each numerator of the original equation by the corresponding complement and drop the denominator. The equation becomes

$$3(n-3) - 2(4n+20) = 6 \cdot 4.$$

5) Solve the above equation:

$$3n-9-8n-40 = 24 \implies -5n-49 = 24 \implies -5n = 24+49,$$

$$-5n = 73 \implies n = -\frac{73}{5}.$$

6) None of the denominators of the original equation is zero for this value of n (check this yourself); so, it is the solution.

In conclusion consider an equation that has the form of equality of two fractions. Such an equation is called a **proportion**. Of course, it can be solved in the same way as before, using LCD. In some cases, it is more convenient to use an important property of proportion: the **cross-multiplication rule**. This rule means the following:

$$\text{if } \frac{a}{b} = \frac{c}{d} \text{ then } ad = bc.$$

In words: the product along one diagonal (a times d) is equal to the product along another diagonal (b times c).

Example 5.7. Solve the equation $\dfrac{5}{2x-4} = \dfrac{2}{3x+5}$.

Solution. The equation is written in the form of a proportion, and we can use the cross-multiplication rule. We have

$$5(3x+5) = 2(2x-4),$$

$$15x+25 = 4x-8 \;\Rightarrow\; 15x-4x = -8-25 \;\Rightarrow\; 11x = -33 \;\Rightarrow\; x = -3.$$

None of the denominators of the original equation is zero for this value of x, so the final answer is $x = -3$.

Exercises 5

In exercises 5.1 – 5.20, solve the given equations.

5.1. $\dfrac{x-2}{8} = \dfrac{5}{8} - \dfrac{x+3}{6}$

5.2. $\dfrac{x+2}{6} = \dfrac{7}{6} - \dfrac{x-5}{9}$

5.3. $\dfrac{2}{4x} + \dfrac{3}{4} = \dfrac{5}{6x}$

5.4. $\dfrac{7}{8x} + \dfrac{5}{12} = \dfrac{3}{8x}$

5.5. $\dfrac{8}{x-4} - 3 = \dfrac{3}{x-4}$

5.6. $\dfrac{9}{x-6} - 4 = \dfrac{5}{x-6}$

5.7. $\dfrac{7}{4} + \dfrac{5}{x-3} = 2$

5.8. $\dfrac{7}{2} + \dfrac{6}{x-5} = 4$

5.9. $\dfrac{3}{x} - \dfrac{7}{x-4} = \dfrac{6}{x}$

5.10. $\dfrac{7}{x} - \dfrac{3}{x-5} = \dfrac{2}{x}$

5.11. $\dfrac{2x}{x-9} - \dfrac{4}{x^2-9x} = 2$

5.12. $\dfrac{3x}{x-4} - \dfrac{2}{x^2-4x} = 3$

5.13. $\dfrac{4}{5} + \dfrac{7}{x^2-9} = \dfrac{4x}{5x-15}$

5.14. $\dfrac{5}{7} + \dfrac{6}{x^2-4} = \dfrac{5x}{7x+14}$

5.15.

$\dfrac{x}{x-4} + \dfrac{1}{x+5} = \dfrac{40-x}{x^2+x-20}$

5.16.

$\dfrac{x}{x-6} + \dfrac{1}{x-2} = \dfrac{30-x}{x^2-8x+12}$

5.17.

$\dfrac{9}{5m+15} - \dfrac{4}{m-2} = \dfrac{7}{m^2+m-6}$

5.18.

$\dfrac{8}{3m+12} - \dfrac{5}{m-3} = \dfrac{7}{m^2+m-12}$

5.19. $\dfrac{2}{7x-9} = \dfrac{3}{4x+6}$

5.20. $\dfrac{4}{5x+3} = \dfrac{7}{8x-3}$

Challenge Problems

5.21. Consider the equation

$$\dfrac{x}{x+a} + \dfrac{1}{x+b} = \dfrac{a-b}{(x+a)(x+b)}$$

Prove the following statements

    a) If $a = 1$ or $b = 1$, then the equation does not have solutions.

    b) If $a \neq 1$ and $b \neq 1$, then the equation has the only solution $x = -1$.

5.22. Consider the equation

$$\frac{x}{x+a} + \frac{1}{x+b} = \frac{a(a-b)}{(x+a)(x+b)}$$

Prove the following statements

    a) If $a = 1$ or $b = 2a - 1$, then the equation does not have solutions.

    b) If $a \neq 1$ and $b \neq 2a - 1$, then the equation has the only solution $x = a - b - 1$.

5.23. Consider the equation

$$\frac{x}{x+a} + \frac{1}{x+b} = \frac{a - 2b - x}{(x+a)(x+b)}$$

Prove the following statements

    a) If $a = 2$ or $b = 2$, then the equation does not have solutions.

    b) If $a \neq 2$ and $b \neq 2$, then the equation has the only solution $x = -2$.

5.24. Consider the equation

$$\frac{x}{x+a} + \frac{1}{x+b} = \frac{a(a-b-1) - x}{(x+a)(x+b)}$$

Prove the following statements

    a) If $a = 2$ or $b = 2a - 2$, then the equation does not have solutions.

    b) If $a \neq 2$ and $b \neq 2a - 2$, then the equation has the only solution
$x = a - b - 2$.

In problems 5.25 – 5.28, you may use the following

Hint: m grams of a% solution contains $\dfrac{ma}{100}$ grams of pure substance.

5.25. A solution of antifreeze contains 20% alcohol. How much pure alcohol must be added to 6 gallons of the solution to make a 40% solution?

5.26. How many gallons of a 15% sugar solution must be mixed with 6 gallons of a 40% sugar solution to make a 30% sugar solution?

5.27. A chemist mixed 4 liters of 18% acid solution with 8 liters of 45% acid solution What percent of acid is in the mixture?

5.28. Nick mixed 9 oz of apple drink with 8 oz of 48% carrot drink. Find the percent of pure apple juice in the apple drink if the mixture contained 30% fruit juice.

Session 6

Radicals and Fractional Exponents

Definition of Radicals

Suppose we want to construct a box in the shape of a cube having the volume of 8 cm^3. The problem is to find its dimensions (i.e. the length of its edges). If we denote this length by x, then the volume is x^3. So, to find x, we need to solve the equation $x^3 = 8$. Perhaps you already recognized that $2^3 = 8$, so $x = 2$ cm is the solution. But what if the volume of a cube is 6? In this case we need to solve the equation $x^3 = 6$. Similar equations may appear in different problems, and we need a way to refer to numbers that their cubes (or other powers) are equal to given numbers. It would be a good idea to invent a special notation for their solutions. Let's consider more general equation $x^n = a$, in which a and n are given (here n is a natural number), and we need to find x. The following symbol was invented to describe the solution of this equation: $\sqrt[n]{}$. This symbol is called the **radical** or **n-th root** or **root of the degree n**. Using it, the solution of the equation $x^n = a$ can be written as $x = \sqrt[n]{a}$, so $\left(\sqrt[n]{a}\right)^n = a$. We may say that to find x, we take the nth root of the number a.

The number n is called the **degree** or **order** of the root. For example, we can read $\sqrt[3]{8}$ as "root of the 3rd order of 8", or "3rd root of 8", or "cube root of 8" (based on the above example with the volume of a cube). By definition, $\sqrt[3]{8} = 2$. For the solution of the equation $x^3 = 6$, we write $x = \sqrt[3]{6}$.

Note. One might think that $\sqrt[3]{6}$ does not exist. However, this is true only if we think about integers. Actually, the number $\sqrt[3]{6}$ exists, but it is neither an integer nor a fraction (rational number). The existence of this number is clearly seen if we want to find the length of a side for a cube with the volume is 6 (such cube exists, so its sides have some length). Numbers like $\sqrt[3]{6}$ are called **irrational** numbers and they cannot be written as fractions. If we want to find such a number as a decimal, we can only get its approximation accurate to a certain number of digits after decimal point. For example, using a calculator, we can find that $\sqrt[3]{6} \approx 1.8$ or $\sqrt[3]{6} \approx 1.82$ and so on. Simple expression $\sqrt[3]{6}$ replaces all possible approximations with any number of digits, and we say that this is the "exact value".

The case when the degree is $n = 2$ is of special interest as it appears most often. For example, assume that we want to construct a square with the area of 9, and we are interested in the length of its side. If we denote this length by x, then the area is x^2. To find x, we need to solve the equation $x^2 = 9$. And the solution is $x = \sqrt[2]{9} = 3$. It is the convention to drop number 2 in the radical $\sqrt[2]{}$, and we simply write $\sqrt{}$. So, $\sqrt{9} = 3$. We read the expression $\sqrt{9}$ as "radical 9" or "square root of 9" (based on the example with the area of a square).

Note. Formally (algebraically) speaking, the equation $x^2 = 9$ has two solutions: $x = 3$ and

$x = -3$ since the square of both numbers is 9. However, by definition, the radical $\sqrt{}$ always refers to a **nonnegative** number, so $\sqrt{9} = 3$, not –3. Both solutions of the equation $x^2 = 9$ can be written as $x = \sqrt{9} = 3$ and $x = -\sqrt{9} = -3$.

Similarly, if a is a nonnegative number, by $\sqrt[n]{a}$ we always mean a nonnegative number.

Note. If we have the expression $\sqrt{x^2}$ and is it not given that x is a nonnegative number, we may **not** always write that $\sqrt{x^2} = x$ since this equality is wrong when x is negative. In this case $\sqrt{x^2} = -x$ (note that $-x$ is positive if x is negative). For example, $\sqrt{(-3)^2} = -(-3) = 3$. The correct equality, which is true for any x, is this:

$$\sqrt{x^2} = |x|,$$

where $|x|$ is the absolute value of x. In the above example, $\sqrt{(-3)^2} = |-3| = 3$. In most parts of this textbook, we will assume that variables (letters) are nonnegative numbers, so we will omit absolute values.

Now, let's give the formal definition of the nth root.

Definition. Let a be a nonnegative number, and n be a natural number (positive integer). Then the nth root of a, denoted by $\sqrt[n]{a}$, is the **nonnegative** solution of the equation $x^n = a$.

In other words, $\sqrt[n]{a}$ is the nonnegative number whose nth power is a:

$$\left(\sqrt[n]{a}\right)^n = a.$$

Number a, which is inside the radical $\sqrt[n]{}$, is called the **radicand**.

Note. We defined $\sqrt[n]{a}$ only for nonnegative a. But what if a is a negative? In this case, we can also define $\sqrt[n]{a}$, but only if the degree n is an **odd** number. For example, $\sqrt[3]{-8} = -2$, since $(-2)^3 = -8$. If n is **even** and a is **negative**, then n^{th} root $\sqrt[n]{a}$ **does not exist** as a real number. For example, $\sqrt{-9}$ does not exist because there is no (real) number x such that $x^2 = -9$.

If the square root of a number is an integer, we call such number a **perfect square**. For example, 9 is a perfect square, but 8 is not. To get a list of all perfect squares, we can take a list of integers 0, 1, 2, 3, …., and square these numbers. We will get the list of perfect squares: 0, 1, 4, 9, …

Properties of Radicals

Similar to exponential expressions, we can multiply and divide radical expressions with the same degree very easily. We will assume in this session that a and b are any nonnegative

numbers, and n is any positive integer.

Product Rule: $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

So, to multiply radical expressions of the same degree, just combine them in one.

Proof. It is enough to show that the nth power of the left and right sides of the above equation is the same. By the definition of the nth root, $\left(\sqrt[n]{a}\right)^n = a$, $\left(\sqrt[n]{b}\right)^n = b$, and $\left(\sqrt[n]{ab}\right)^n = ab$. Using Product Rule for exponents (see Session 3), we have

$$\left(\sqrt[n]{a} \times \sqrt[n]{b}\right)^n = \left(\sqrt[n]{a}\right)^n \times \left(\sqrt[n]{b}\right)^n = ab = \left(\sqrt[n]{ab}\right)^n .$$

Example 6.1. Multiply and simplify $\sqrt{2} \times \sqrt{18}$.

Solution. $\sqrt{2} \times \sqrt{18} = \sqrt{2 \times 18} = \sqrt{36} = 6.$

Quotient Rule: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$, $b \neq 0.$

As you see, similar to the Product Rule, we can combine two radical expressions of the same degree into one. The proof is also similar.

Note. Informally, you may think about radicals as umbrellas above the numbers. Product and Quotient Rules say that you may replace two umbrellas with one that covers both numbers.

Example 6.2. Divide and simplify $\dfrac{\sqrt{50}}{\sqrt{2}}$.

Solution. $\dfrac{\sqrt{50}}{\sqrt{2}} = \sqrt{\dfrac{50}{2}} = \sqrt{25} = 5.$

Below we will show how to combine radical expressions of different degrees.

Power Rule: $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ for positive integer m.

In particular, $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$ (actually, this is the definition of the radical).

Proof. By the definition of the mth power and the above Product Rule,

$$\left(\sqrt[n]{a}\right)^m = \underbrace{\sqrt[n]{a} \times \sqrt[n]{a} \times ... \times \sqrt[n]{a}}_{m \text{ copies}} = \sqrt[n]{a \times a \times ... \times a} = \sqrt[n]{a^m}$$

Example 6.3. Simplify $\left(\sqrt{2}\right)^4$.

Solution. $\left(\sqrt{2}\right)^4 = \sqrt{(2)^4} = \sqrt{16} = 4.$

Fractional Exponents

So far, we multiply and divide radical expressions of the same degree only. But what if we need to operate with different degrees? For example, is it possible to somehow represent the product $\sqrt{2} \times \sqrt[3]{2}$ using only one radical? Here we learn how to do this. The idea is to set up a direct connection between radicals and exponents. In this way, we could apply product, quotient and power rules of exponents to radicals.

Let's try to represent radical expression $\sqrt[n]{a}$ as exponential with the base a and some power m: $\sqrt[n]{a} = a^m$. If we raise both sides of this equation to the nth power, we will have

$$\left(\sqrt[n]{a}\right)^n = \left(a^m\right)^n = a^{m \cdot n} \text{. But } \left(\sqrt[n]{a}\right)^n = a = a^1 \text{. Therefore, } a^1 = a^{m \cdot n} \text{.}$$

From here we can equate powers: $1 = mn$, and $m = \dfrac{1}{n}$. Now the expression $\sqrt[n]{a} = a^m$ can

be written as $\sqrt[n]{a} = a^{\frac{1}{n}}$. We were able to present the nth root as an exponential expression

with the fractional power $\dfrac{1}{n}$! We use this representation as a definition.

Definition. Let a be any nonnegative number, and n be any positive integer. Then

$$a^{\frac{1}{n}} = \sqrt[n]{a} \text{.}$$

In particular, $a^{\frac{1}{2}} = \sqrt{a}$.

We can easily generalize this definition to exponential expressions with arbitrary fractional

power $\dfrac{m}{n}$. To do this, just raise both sides of the equation $a^{\frac{1}{n}} = \sqrt[n]{a}$ to the mth power and

use the power rules for exponents and radicals:

$$\left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \implies a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{.}$$

This motivates the following

Definition. Let a be any non-negative number, and $\dfrac{m}{n}$ be any positive fraction. Then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{.}$$

Note. Be careful to put m and n on the right side in the correct places: the numerator m is the power of a, and the denominator n is the degree of the radical.

Representing radicals as fractional exponents expands our ability to manipulate with radicals. Here is an example of how it may help to simplify radical expressions of different degrees.

Example 6.4. Simplify the expression $\sqrt{2} \times \sqrt[3]{2}$ (combine using one radical).

Solution. $\sqrt{2} \times \sqrt[3]{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{2} + \frac{1}{3}} = 2^{\frac{3+2}{6}} = 2^{\frac{5}{6}} = \sqrt[6]{2^5} = \sqrt[6]{32}$.

Note. In solving the above example, we used the Product Rule. This rule was given in Session 3 for integer exponents only. Here is a possible way to justify this rule for fractional exponents:

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \cdot a^{\frac{pn}{qn}} = \sqrt[nq]{a^{mq}} \cdot \sqrt[nq]{a^{pn}} = \sqrt[nq]{a^{mq} \cdot a^{pn}}$$

$$= \sqrt[nq]{a^{mq+pn}} = a^{\frac{mq+pn}{nq}} = a^{\frac{m}{n}+\frac{p}{q}}.$$

In session 3 we considered exponential expressions with negative integer exponents. So far in this session, we have dealt with positive fractional exponents. Now we can combine these two types of exponents together using the corresponding formulas

$$a^{-k} = \frac{1}{a^k} \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m} \ .$$

From here we see the connection between **negative fractional exponents** and **radicals**:

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}}, \quad a > 0.$$

Note. In the definition of the expression $a^{\frac{m}{n}}$ with fractional exponent $\dfrac{m}{n}$, we always assume that the base a is a positive number. The reason is to avoid ambiguous situation that may arise when a is negative. For example, compare the following two expressions

$$(-1)^{\frac{1}{3}} \text{ and } (-1)^{\frac{2}{6}}.$$

Since $\dfrac{1}{3} = \dfrac{2}{6}$, it is reasonable to expect that both expressions are equal. However,

$$(-1)^{\frac{1}{3}} = \sqrt[3]{-1} = -1, \text{ but } (-1)^{\frac{2}{6}} = \sqrt[6]{(-1)^2} = \sqrt[6]{1} = 1.$$

Example 6.5. Simplify the expression $\dfrac{\sqrt[10]{3^7}}{\sqrt[5]{3}}$.

Solution. $\dfrac{\sqrt[10]{3^7}}{\sqrt[5]{3}} = \dfrac{3^{\frac{7}{10}}}{3^{\frac{1}{5}}} = 3^{\frac{7}{10}-\frac{1}{5}} = 3^{\frac{7-2}{10}} = 3^{\frac{5}{10}} = 3^{\frac{1}{2}} = \sqrt{3}$.

Simplification of Square Roots

Here we focus on square roots, but a similar technique can also be used for general exponents. By simplification, we mean the modification of a given expression in such a way that the smallest possible expression remains inside the radical (simplest radicand).

1. Simplification of the numerical expression $\sqrt{a}$ (a is a number).

Session 6: Radicals and Fractional Exponents

Let's re-write the Product Rule, described above, from right to left: $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$. In this way, we may split one radical expression into a product of two. In particular, $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$. This property helps to simplify some radical expressions.

Example 6.6. Simplify $\sqrt{12}$.

Solution. We split 12 as 4×3. Then $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$.

Note. We can also split 12 in a different way: $12 = 2 \times 6$. However, this way will not lead to simplification since both factors, 2 and 6, are not perfect squares. Try to split the given expression in such a way that one of the factors is a perfect square.

Example 6.7. Simplify $\sqrt{48}$.

Solution. It is possible to split number 48 as a product of two factors, one of which is a perfect square, in two ways: $48 = 4 \times 12$ and $48 = 16 \times 3$. Both options work. However, if we use the first way, we need to continue factoring 12 as 4×3, and this way is longer. The second way is shorter. In general, try to split the given number in such a way that a factor which is not a perfect square cannot be factored further to contain another perfect square. So, in this example we choose the second factorization $48 = 16 \times 3$ since number 3 has no perfect square factor. Using this, we get $\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$.

In sections 2 and 3 below, we assume that $x \geq 0$.

2. Simplification of $\sqrt{x^n}$ for even *n*.

We can represent the square root using an exponent of $\dfrac{1}{2}$, and then use the power rule:

$$\sqrt{x^n} = \left(x^n\right)^{\frac{1}{2}} = x^{\frac{n}{2}}.$$

We come up to the following method of simplifying the square root from x^n with the even power *n*: **divide power by 2** and remove radical.

Example 6.8. Simplify $\sqrt{x^{10}}$.

Solution. According to the above rule, $\sqrt{x^{10}} = x^{\frac{10}{2}} = x^5$.

Example 6.9. Simplify $\sqrt{x^{16}}$.

Solution. According to the above rule, $\sqrt{x^{16}} = x^{\frac{16}{2}} = x^8$.

Note. Don't be misled that in the above example power 16 is a perfect square, so one might think of taking the square root of 16. **Do not** take the square root of the exponent, instead, **divide** the exponent by two.

3. Simplification of $\sqrt{x^n}$ for odd *n*.

Any odd number can be written as $2m + 1$, where *m* is an integer, so we can write $\sqrt{x^n}$ as

$\sqrt{x^{2m+1}}$. To simplify this expression, we represent x^{2m+1} as $x^{2m} \cdot x$ (separate an even power and a single x). Then

$$\sqrt{x^{2m+1}} = \sqrt{x^{2m} \cdot x} = \sqrt{x^{2m}} \cdot \sqrt{x} = x^m \sqrt{x}.$$

We come up to the following rule, how to simplify the square root of x^{2m+1}: detach (separate) single x from x^{2m+1}: $x^{2m+1} = x^{2m} \cdot x$, then leave x inside the radical and take the square root from x^{2m} which is x^m.

Example 6.10. Simplify $\sqrt{x^7}$.

Solution. Using the above rule, $\sqrt{x^7} = \sqrt{x^6 \cdot x} = \sqrt{x^6} \cdot \sqrt{x} = x^3 \sqrt{x}$.

Example 6.11. Simplify $\sqrt{x^{25}}$.

Solution. Using the above rule, $\sqrt{x^{25}} = \sqrt{x^{24} \cdot x} = \sqrt{x^{24}} \cdot \sqrt{x} = x^{12} \sqrt{x}$.

Similar to the above note for the even power, **do not** take square root from 25.

Example 6.12. Simplify $\sqrt{9x^9}$.

Solution. Using the above rule, $\sqrt{9x^9} = \sqrt{9x^8 \cdot x} = 3x^4 \sqrt{x}$.

Note that the number 9 is used here in two different ways depending on whether it is a coefficient next to a variable or an exponent: take square root from the coefficient but not from the power.

In conclusion, consider an example that combines all three types of radicals described above: numbers, and exponential expressions with even and odd powers of variables.

Example 6.13. Simplify the expression $4 \cdot \sqrt{150x^{12}y^7}$.

Solution.

1st method. (Simplify each factor separately). The radicand is a product of three factors: n 150, x^{12} and y^7. We can simplify each of them separately:

$$\sqrt{150} = \sqrt{25 \times 6} = 5\sqrt{6},$$

$$\sqrt{x^{12}} = x^6,$$

$$\sqrt{y^7} = \sqrt{y^6 \cdot y} = y^3 \sqrt{y}.$$

From here, $4 \cdot \sqrt{150x^{12}y^7} = 4 \cdot \sqrt{150} \cdot \sqrt{x^{12}} \cdot \sqrt{y^7} = 4 \cdot 5 \cdot \sqrt{6} \cdot x^6 \cdot y^3 \sqrt{y} = 20x^6y^3\sqrt{6y}$.

2nd method. (Simplify the entire expression at once).

$$4 \cdot \sqrt{150x^{12}y^7} = 4 \cdot \sqrt{25 \cdot 6 \cdot x^{12} \cdot y^6 \cdot y} = 4 \cdot 5 \cdot x^6 \cdot y^3 \sqrt{6 \cdot y} = 20x^6y^3\sqrt{6y}.$$

Exercises 6

In exercises 6.1 and 6.2, evaluate the given expressions without using a calculator.

6.1. a) $\sqrt{2} \times \sqrt{8}$

   b) $\dfrac{\sqrt{27}}{\sqrt{3}}$

   c) $\left(\sqrt{5}\right)^4$

   d) $16^{\frac{1}{2}}$

   e) $8^{-\frac{1}{3}}$

   f) $25^{\frac{3}{2}}$

   g) $27^{-\frac{2}{3}}$

   h) $\sqrt[3]{4} \times \sqrt[6]{4}$

6.2. a) $\sqrt{3} \times \sqrt{27}$

   b) $\dfrac{\sqrt{80}}{\sqrt{5}}$

   c) $\left(\sqrt{6}\right)^4$

   d) $27^{\frac{1}{3}}$

   e) $36^{-\frac{1}{2}}$

   f) $8^{\frac{2}{3}}$

   g) $4^{-\frac{3}{2}}$

   h) $\sqrt[6]{81} \times \sqrt[12]{81}$

In exercises 6.3 and 6.4, write the given expressions using one radical symbol and simplify.

6.3. $\dfrac{\sqrt[7]{a}}{\sqrt[14]{a^9}}$

6.4. $\dfrac{\sqrt[6]{b^5}}{\sqrt[3]{b}}$

In exercises 6.5 – 6.14, simplify the given expressions.

6.5. $\sqrt{50}$

6.6. $\sqrt{54}$

6.7. $\sqrt{32}$

6.8. $\sqrt{72}$

6.9. a) $\sqrt{x^8}$

   b) $\sqrt{x^7}$

6.10. a) $\sqrt{x^6}$

   b) $\sqrt{x^5}$

6.11. a) $\sqrt{36y^{36}}$

   b) $\sqrt{9y^9}$

6.12. a) $\sqrt{49z^{49}}$

   b) $\sqrt{64z^{64}}$

6.13. $3 \cdot \sqrt{75x^{10}y^5z^{16}}$

6.14. $5 \cdot \sqrt{75x^{14}y^9z^{13}}$

In exercises 6.15 and 6.16, simplify and write your answers using positive exponents only.

6.15. $\left(\dfrac{a^{-3/4}}{b^{-4/3}}\right)^{12}$

6.16. $\left(\dfrac{b^{-5/3}}{a^{-3/5}}\right)^{15}$

Challenge Problems

6.17. Calculate without using a calculator

$$\sqrt[3]{\dfrac{5^{18}+5^{30}}{5^{24}+5^{12}}}$$

6.18. Prove that

$$\sqrt[n]{\dfrac{a^m+a^{m+p}}{a^k+a^{k+p}}}=a^{\frac{m-k}{n}}$$

Session 7

Multiplication, Addition and Subtraction of Radical Expressions

Multiplication of Radical Expressions

In previous session we considered the Product Rule for radical expressions of the same degree n: $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$. This rule allows to combine a product of two (or more) radical expressions into one. Here we consider more examples for multiplication of radical expressions. We restrict ourselves to square roots only. Also, we assume that all letters represent nonnegative numbers.

Let's recall that $\sqrt{a \cdot a} = a$. This simple formula allows us to avoid tedious calculations in some cases: if you notice inside the radical repeated factors, do not multiply them, just extract the common factor from the radical.

Example 7.1. Multiply and simplify $\sqrt{23} \cdot \sqrt{46}$.

Solution. One way to proceed is to directly multiply the radicands:

$$\sqrt{23} \cdot \sqrt{46} = \sqrt{23 \cdot 46} = \sqrt{1058}.$$

Now you need to simplify $\sqrt{1058}$. Even though it is possible, this is not the best way since it might be difficult to continue. Notice, however, that $46 = 23 \cdot 2$, and it is much easier to proceed like this:

$$\sqrt{23} \cdot \sqrt{46} = \sqrt{23} \cdot \sqrt{23 \cdot 2} = \sqrt{23 \cdot 23 \cdot 2} = 23\sqrt{2}.$$

Example 7.2. Multiply and simplify $\left(5\sqrt{7x^5 y}\right)\left(3\sqrt{14x^3 y^4}\right)$

Solution. We multiply separately numbers which are outside the radicals (5 and 3) and radicands (we also represent 14 as $7 \cdot 2$):

$$\left(5\sqrt{7x^5 y}\right)\left(3\sqrt{14x^3 y^4}\right) = (5 \cdot 3)\sqrt{\left(7x^5 y\right) \cdot \left(7 \cdot 2x^3 y^4\right)}$$

$$= 15\sqrt{7 \cdot 7 \cdot 2x^8 yy^4} = 15 \cdot 7x^4 y^2 \sqrt{2y} = 105x^4 y^2 \sqrt{2y}$$

Note that the radicand $2y$ doesn't have perfect square factors, so we cannot simplify the expression further.

Addition and Subtraction Radicals

Contrary to the product rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, there is no simple rule to add or subtract

radical: in general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$. Here is an example: $a = 9$ and $b = 16$. Then

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7 \text{, but } \sqrt{9+16} = \sqrt{25} = 5 \text{, so } \sqrt{9} + \sqrt{16} \neq \sqrt{9+16} \text{.}$$

We can add or subtract radicals directly only if the radicands are exactly the same. This procedure is similar to combining like terms.

Example 7.3. Add $5\sqrt{7} + 3\sqrt{7}$.

Solution. Similar to combining like terms $5x + 3x = 8x$, we have $5\sqrt{7} + 3\sqrt{7} = 8\sqrt{7}$.

Example 7.4. Simplify the expression $a\sqrt{5x} + b\sqrt{5x}$.

Solution. $a\sqrt{5x} + b\sqrt{5x} = (a+b)\sqrt{5x}$.

Example 7.5. Simplify the expression $x\sqrt{2y} + 3\sqrt{5z} - 4\sqrt{2y} - x\sqrt{5z}$.

Solution. This expression contains four terms. As we indicated above, we can combine only those terms that have the same radicands. We can combine the first term with the third (they contain the same $\sqrt{2y}$), and the second with the fourth (they contain the same $\sqrt{5z}$):

$$x\sqrt{2y} + 3\sqrt{5z} - 4\sqrt{2y} - x\sqrt{5z} = (x-4)\sqrt{2y} + (3-x)\sqrt{5z} \text{.}$$

Note. As you can see, the final answer contains two radicals. We cannot combine them in one radical because they have different radicands, so no further processing possible.

There are cases when, even if the original expression contains different radicands, it is still possible to combine them. These are the cases when after simplifying the individual radicals, they contain common radicands.

Example 7.6. Simplify the expression

a) $\sqrt{20u^3} + u\sqrt{45u}$.

b) $6\sqrt{8} + 5\sqrt{27} - 4\sqrt{32} - 2\sqrt{75}$

Solution.

a) There are two radicands here and they are different, so we cannot combine them initially. Let's simplify them first (we will process the entire expression):

$$\sqrt{20u^3} + u\sqrt{45u} = \sqrt{4 \cdot 5u^2 u} + u\sqrt{9 \cdot 5u} = 2u\sqrt{5u} + 3u\sqrt{5u} = 5u\sqrt{5u} \text{.}$$

b) There are four radicals here and all of them have different radicands. Similar to part a), we simplify them first:

$$6\sqrt{8}+5\sqrt{27}-4\sqrt{32}-2\sqrt{75}=6\sqrt{4\cdot2}+5\sqrt{9\cdot3}-4\sqrt{16\cdot2}-2\sqrt{25\cdot3}$$
$$=6\cdot2\sqrt{2}+5\cdot3\sqrt{3}-4\cdot4\sqrt{2}-2\cdot5\sqrt{3}=12\sqrt{2}+15\sqrt{3}-16\sqrt{2}-10\sqrt{3}.$$

Now, the first and the third terms contain the same $\sqrt{2}$, and the second and the fourth contain the same $\sqrt{3}$. Therefore, we can combine them:

$$12\sqrt{2}+15\sqrt{3}-16\sqrt{2}-10\sqrt{3}=(12-16)\sqrt{2}+(15-10)\sqrt{3}=-4\sqrt{2}+5\sqrt{3}.$$

It is not possible to combine further since both radicands are different and cannot be simplified more.

Mixed Problems

Example 7.7. Multiply and simplify $3\sqrt{5xy}\left(2\sqrt{15}x-4\sqrt{30y}\right)$.

Solution. We can distribute using the usual distributive property:

$$3\sqrt{5xy}\left(2\sqrt{15}x-4\sqrt{30y}\right)=3\sqrt{5xy}\cdot2\sqrt{15}x-3\sqrt{5xy}\cdot4\sqrt{30y}.$$

Next, we can simplify each term separately:

$$3\sqrt{5xy}\cdot2\sqrt{15}x=3\cdot2\cdot\sqrt{5xy}\cdot\sqrt{15}x=6\sqrt{5xy\cdot15x}=6\sqrt{5\cdot5\cdot3x^2y}=6\cdot5x\sqrt{3y}=30x\sqrt{3y},$$
$$3\sqrt{5xy}\cdot4\sqrt{30y}=3\cdot4\sqrt{5xy}\cdot\sqrt{30y}=12\sqrt{5xy\cdot30y}=12\sqrt{5\cdot5\cdot6xy^2}=12\cdot5y\sqrt{6x}=60y\sqrt{6x}$$

Finally, we subtract the last expression from the previous one and get the answer

$$3\sqrt{5xy}\left(2\sqrt{15}x-4\sqrt{30y}\right)=30x\sqrt{3}y-60y\sqrt{6x}.$$

Example 7.8. Multiply and simplify $\left(3\sqrt{2}-4\right)\left(5\sqrt{3}+\sqrt{6}\right)$.

Solution. As in Example 7.7, we start with distribution:

$$\left(3\sqrt{2}-4\right)\left(5\sqrt{3}+\sqrt{6}\right)=3\sqrt{2}\cdot5\sqrt{3}+3\sqrt{2}\cdot\sqrt{6}-4\cdot5\sqrt{3}-4\sqrt{6}.$$

Again, we can simplify each term separately (if you want, you may simplify them simultaneously):

$$3\sqrt{2}\cdot5\sqrt{3}=3\cdot5\sqrt{2\cdot3}=15\sqrt{6},$$
$$3\sqrt{2}\cdot\sqrt{6}=3\sqrt{2\cdot6}=3\sqrt{2\cdot2\cdot3}=3\cdot2\sqrt{3}=6\sqrt{3}.$$
$$4\cdot5\sqrt{3}=20\sqrt{3}.$$

From here,

$$\left(3\sqrt{2}-4\right)\left(5\sqrt{3}+\sqrt{6}\right)=15\sqrt{6}+6\sqrt{3}-20\sqrt{3}-4\sqrt{6}$$
$$=(15-4)\sqrt{6}+(6-20)\sqrt{3}=11\sqrt{6}-14\sqrt{3}\,.$$

Below, we consider an example that can be easily solved using the following difference of squares formula, which we already mentioned in session 2:

$$(a-b)(a+b)=a^{2}-b^{2}\,.$$

Example 7.9. Multiply and simplify $\left(2\sqrt{5}-3\sqrt{7}\right)\left(2\sqrt{5}+3\sqrt{7}\right)$.

Solution. We can solve this problem in the same way as we did in Example 7.8 by distributing. Notice, however, that inside the first and the second sets of parentheses we have difference and sum of the same expressions $2\sqrt{5}$ and $3\sqrt{7}$. Therefore, we can use the above difference of squares formula with $a=2\sqrt{5}$ and $b=3\sqrt{7}$. We have

$$a^{2}=\left(2\sqrt{5}\right)^{2}=2^{2}\cdot\left(\sqrt{5}\right)^{2}=4\cdot5=20\,,$$

and

$$b^{2}=\left(3\sqrt{7}\right)^{2}=3^{2}\cdot\left(\sqrt{7}\right)^{2}=9\cdot7=63\,.$$

Now, subtract $a^{2}-b^{2}$ and get the answer

$$\left(2\sqrt{5}-3\sqrt{7}\right)\left(2\sqrt{5}+3\sqrt{7}\right)=20-63=-43\,.$$

In the following example we solve a problem similar to Example 7.9 in more general form.

Example 7.10. Multiply and simplify $\left(m\sqrt{n}-x\sqrt{y}\right)\left(m\sqrt{n}+x\sqrt{y}\right)$.

Solution.

$$\left(m\sqrt{n}-x\sqrt{y}\right)\left(m\sqrt{n}+x\sqrt{y}\right)=\left(m\sqrt{n}\right)^{2}-\left(x\sqrt{y}\right)^{2}$$
$$=m^{2}\left(\sqrt{n}\right)^{2}-x^{2}\left(\sqrt{y}\right)^{2}=m^{2}n-x^{2}y.$$

Note. The expressions $m\sqrt{n}-x\sqrt{y}$ and $m\sqrt{n}+x\sqrt{y}$ are called **conjugate** to each other. Example 7.10 shows that the product of conjugate expressions does not contain radicals. In the next session, we will use this property to "rationalize" the denominators.

Exercises 7

In exercises 7.1 – 7.6, multiply and simplify the given expressions.

7.1. a) $\sqrt{7} \cdot \sqrt{7}$ 7.2. a) $\sqrt{5} \cdot \sqrt{5}$

   b) $\sqrt{2018} \cdot \sqrt{2018}$ b) $\sqrt{2019} \cdot \sqrt{2019}$

   c) $\sqrt{47} \cdot \sqrt{94}$ c) $\sqrt{26} \cdot \sqrt{78}$

7.3. $\sqrt{14} \cdot \sqrt{21}$ 7.4. $\sqrt{18} \cdot \sqrt{30}$

7.5. a) $\sqrt{3p^5 q^6} \cdot \sqrt{15 p^3 q^3}$ 7.6. a) $\sqrt{5m^8 n^3} \cdot \sqrt{15 m^3 n^5}$

   b) $\left(4\sqrt{24a^3 b^2}\right) \cdot \left(6\sqrt{72a^7 b}\right)$ b) $\left(2\sqrt{21u^9 v^6}\right) \cdot \left(7\sqrt{42u^3 v}\right)$

   c) $\left(5\sqrt{18r^5 s^4}\right) \cdot \left(3\sqrt{36rs}\right)$ c) $\left(8\sqrt{15p^8 q^{10}}\right) \cdot \left(4\sqrt{25pq^2}\right)$

In exercises 7.7 and 7.8, add or subtract the given expressions.

7.7. a) $3\sqrt{6} + 8\sqrt{6}$ 7.8. a) $7\sqrt{5} + 2\sqrt{5}$

   b) $3\sqrt{7} - 5\sqrt{7}$ b) $5\sqrt{3} - 8\sqrt{3}$

In exercises 7.9 – 7.12, simplify the given expressions.

7.9. $3m\sqrt{6n} - 2\sqrt{7k} + 4\sqrt{7k} + 5m\sqrt{6n}$ 7.10. $7p\sqrt{3q} - 2\sqrt{6r} + 4\sqrt{6r} + 5p\sqrt{3q}$

7.11. a) $\sqrt{8} + \sqrt{50}$ 7.12. a) $\sqrt{12} + \sqrt{27}$

   b) $5\sqrt{27} - 2\sqrt{48}$ b) $6\sqrt{45} - 4\sqrt{80}$

   c) $8\sqrt{24} + 4\sqrt{20} - 2\sqrt{54} - 6\sqrt{45}$ c) $6\sqrt{63} - 7\sqrt{48} + 5\sqrt{28} - 2\sqrt{108}$

In exercises 7.13 – 7.26, multiply and simplify the given expressions.

7.13. $\sqrt{10}\left(\sqrt{2} + \sqrt{5}\right)$ 7.14. $\sqrt{6}\left(\sqrt{2} + \sqrt{3}\right)$

7.15. $5\sqrt{7}\left(4\sqrt{63} - 3\sqrt{7}\right)$ 7.16. $4\sqrt{3}\left(6\sqrt{3} - 2\sqrt{12}\right)$

7.17. $\left(2\sqrt{5} + 3\sqrt{6}\right)\left(4\sqrt{5} - 2\sqrt{6}\right)$ 7.18 $\left(5\sqrt{2} - 4\sqrt{7}\right)\left(3\sqrt{2} + 2\sqrt{7}\right)$

7.19. $\left(3\sqrt{6} - 2\sqrt{3}\right)\left(3\sqrt{6} + 2\sqrt{3}\right)$ 7.20. $\left(4\sqrt{7} + 3\sqrt{5}\right)\left(4\sqrt{7} - 3\sqrt{5}\right)$

7.21. $\left(5\sqrt{5}+2\sqrt{2}\right)\left(5\sqrt{5}-2\sqrt{2}\right)$

7.22. $\left(6\sqrt{6}-3\sqrt{3}\right)\left(6\sqrt{6}+3\sqrt{3}\right)$

7.23. $\left(3\sqrt{7}+2\sqrt{5}\right)^{2}$

7.24. $\left(2\sqrt{6}+4\sqrt{3}\right)^{2}$

7.25. $\left(3\sqrt{7}-2\sqrt{5}\right)^{2}$

7.26. $\left(2\sqrt{6}-4\sqrt{3}\right)^{2}$

Challenge Problems

7.27. Prove that $\sqrt{5+\sqrt{24}}=\sqrt{2}+\sqrt{3}$.

7.28. Prove that for any nonnegative number a, $\sqrt{a+1+\sqrt{4a}}-\sqrt{a}=1$.

7.29. Let's call a square root simple if there are no other square roots inside it. For example, $\sqrt{3}$ is a simple square root, but $\sqrt{3+\sqrt{8}}$ is not (such expression is also called nested root). Represent the expression $\sqrt{3+\sqrt{8}}-1$ as a simple square root.

Session 8

Rationalizing the Denominators and Solving Radical Equations

Rationalizing the Denominators

Similar to the product rule $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, we can use the following quotient rule to divide radicals (we consider here square roots only): $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$. As with the product rule, this rule allows to replace two "umbrellas" (two radicals) for a and b, with one "umbrella" that covers both. For example, $\dfrac{\sqrt{10}}{\sqrt{6}} = \sqrt{\dfrac{10}{6}} = \sqrt{\dfrac{5}{3}}$.

Also, we can do just the opposite: brake one radical into two: $\sqrt{\dfrac{5}{3}} = \dfrac{\sqrt{5}}{\sqrt{3}}$.

In some cases, it is desirable to modify expressions like this further to get rid of the radical in the denominator. The procedure for doing this is called **rationalization the denominator**.

The general idea to rationalize the denominator is to use the main property of a fraction: if we multiply the numerator and denominator of a fraction by the same nonzero expression, the value of the fraction remains the same (even if the fraction will look different). We consider here two types of fractions: one with a single term with a radical in the denominator, and another with a sum (or difference) of two terms in the denominator (where at least one of them contains radical).

Fractions with a single radical term in the denominator.

Such fractions have the following general form $\dfrac{\text{expr}}{m\sqrt{n}}$, where expr means some expression.

To rationalize the denominator, we multiply both the numerator and denominator by $\sqrt{n}$ and simplify the denominator, using the property $\sqrt{n} \cdot \sqrt{n} = n$. In doing this, we get rid of the radical in the denominator (so, we rationalize the denominator):

$$\frac{\text{expr}}{m\sqrt{n}} = \frac{\text{expr}\sqrt{n}}{m\sqrt{n}\sqrt{n}} = \frac{\text{expr}\sqrt{n}}{mn}.$$

Example 8.1. Rationalize the denominator: $\sqrt{\dfrac{5}{3}}$.

Solution. We have $\sqrt{\dfrac{5}{3}} = \dfrac{\sqrt{5}}{\sqrt{3}}$. To continue (to get rid of $\sqrt{3}$ in the denominator), we multiply the numerator and the denominators by $\sqrt{3}$:

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{15}}{3}.$$

Example 8.2. Rationalize the denominator: $\dfrac{6}{5\sqrt{8}}$.

Solution. Following the same method, we multiply the numerator and denominator by $\sqrt{8}$ (it is not needed to also multiply by 5):

$$\frac{6}{5\sqrt{8}} = \frac{6 \cdot \sqrt{8}}{5\sqrt{8} \cdot \sqrt{8}} = \frac{6\sqrt{8}}{5 \cdot 8} = \frac{6\sqrt{4 \cdot 2}}{5 \cdot 8} = \frac{6 \cdot 2\sqrt{2}}{5 \cdot 8} = \frac{3\sqrt{2}}{10}.$$

Here we also simplified $\sqrt{8}$. If you compare initial fraction $\dfrac{6}{5\sqrt{8}}$ with the final answer $\dfrac{3\sqrt{2}}{10}$, they seem completely different. However, they have exactly the same numerical values (you can check this using a calculator).

Fractions with the sum or difference of two terms in the denominator.

Such fractions have the following general form

$$\frac{\text{expr}}{m\sqrt{n} - x\sqrt{y}} \quad \text{or} \quad \frac{\text{expr}}{m\sqrt{n} + x\sqrt{y}},$$

where expr, as before, means some expression. The denominators in these two fractions are **conjugate** to each other. As we already saw in example 7.10 from the previous session, their product is a rational expression (i.e. an expression with no radicals):

$$\left(m\sqrt{n} - x\sqrt{y}\right)\left(m\sqrt{n} + x\sqrt{y}\right) = m^2 n - x^2 y.$$

This property allows us to rationalize the denominators: multiply the numerator and denominator by the expression conjugate to the denominator.

Note. If you try to multiply the numerator and denominator of the above fractions by only one of the radicals (for example, by $\sqrt{n}$), you will still have radicals in denominator. So, do not confuse the two cases: single term in the denominator (when we multiply the numerator and denominator by the radical in the denominator) and the sum or difference of two terms (when we multiply the numerator and denominator by the expression conjugate to the entire denominator). For example, consider two fractions:

$$\frac{a}{b\sqrt{c}} \quad \text{and} \quad \frac{a}{b+\sqrt{c}}\,.$$

We use **different** methods to rationalize them: for the first fraction we multiply the numerator and denominator by $\sqrt{c}$, while for the second fraction we multiply the numerator and denominator by $b - \sqrt{c}$:

$$\frac{a}{b\sqrt{c}} = \frac{a\sqrt{c}}{b\sqrt{c}\cdot\sqrt{c}} = \frac{a\sqrt{c}}{bc}, \qquad \frac{a}{b+\sqrt{c}} = \frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})} = \frac{a(b-\sqrt{c})}{b^2-c}\,.$$

Example 8.3. Rationalize the denominator: $\dfrac{1}{2-\sqrt{3}}\,.$

Solution. Here we have a difference of two terms in the denominator: $2-\sqrt{3}$. Therefore, we multiply the numerator and denominator by the expression conjugate to the denominator. The conjugate expression is $2+\sqrt{3}$.

$$\frac{1}{2-\sqrt{3}} = \frac{1\cdot(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2+\sqrt{3}}{2^2-(\sqrt{3})^2} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}\,.$$

Final answer looks pretty nice: $\dfrac{1}{2-\sqrt{3}} = 2+\sqrt{3}\,.$

Example 8.4. Rationalize the denominator: $\dfrac{\sqrt{x}-3}{3\sqrt{x}+2\sqrt{y}}\,.$

Solution. The expression conjugate to the denominator is $3\sqrt{x}-2\sqrt{y}$. We multiply the numerator and denominator by it:

$$\frac{\sqrt{x}-3}{3\sqrt{x}+2\sqrt{y}} = \frac{(\sqrt{x}-3)(3\sqrt{x}-2\sqrt{y})}{(3\sqrt{x}+2\sqrt{y})(3\sqrt{x}-2\sqrt{y})} = \frac{\sqrt{x}\cdot3\sqrt{x}-2\sqrt{x}\cdot\sqrt{y}-3\cdot3\sqrt{x}+3\cdot2\sqrt{y}}{3^2(\sqrt{x})^2-2^2(\sqrt{y})^2}$$

$$= \frac{3x-2\sqrt{xy}-9\sqrt{x}+6\sqrt{y}}{9x-4y}\,.$$

Solving Radical Equations

We now consider equations that contain radicals. They can be transformed into equations without radicals in two steps:

Step 1: If needed, isolate the term with the radical (i.e. leave it alone on one side of the equation).

Step 2: Square both sides of the equation.

Note. When you square both sides of an equation, it is possible to get so-called "extraneous solutions", that is, answers that are not solutions to the original equation. Here is a simple example: $x + 1 = 2$. Obviously, this equation has only one solution $x = 1$. Now, if we square both sides of this equation, we get $(x+1)^2 = 2^2$, or

$$x^2 + 2x + 1 = 4 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow (x-1)(x+3) = 0.$$

The last equation has two solutions: $x = 1$ and $x = -3$. However, the value $x = -3$ is not a solution to the original equation $x + 1 = 2$ (it is the extraneous one). The conclusion from this note is this: always check your final answers with the original equation.

Example 8.5. Solve the equation $\sqrt{2x-1} = 7$.

Solution. Here, the radical is already isolated, and the first step is not needed. We just square both sides:

$$\left(\sqrt{2x-1}\right)^2 = 7^2 \Rightarrow 2x - 1 = 49 \Rightarrow 2x = 49 + 1 \Rightarrow 2x = 50 \Rightarrow x = 25 .$$

It is easy to verify that the number 25 is a solution of the original equation:

$\sqrt{2 \cdot 25 - 1} = \sqrt{49} = 7$. Final answer: $x = 25$.

Example 8.6. Solve the equation $\sqrt{6x-5} + 7 = 6$.

Solution. Here the radical is not isolated, so we isolate it by moving number 7 to the right side:

$$\sqrt{6x-5} = 6 - 7 \text{ or } \sqrt{6x-5} = -1.$$

 Now, the radical is isolated, and we square both sides:

$$\left(\sqrt{6x-5}\right)^2 = (-1)^2 \Rightarrow 6x - 5 = 1 \Rightarrow 6x = 1 + 5 \Rightarrow 6x = 6 \Rightarrow x = 1 .$$

So, it looks like $x = 1$ is a solution. Let's check it with the original equation. Calculate the left side:

$$\sqrt{6 \cdot 1 - 5} + 7 = \sqrt{1} + 7 = 1 + 7 = 8 .$$

 However, the right side of the original equation is 6, not 8, so $x = 1$ is not a solution and we reject it. The original equation has no solutions at all.

 Note. It can be seen at the very beginning without doing anything that the original equation doesn't have solutions. Indeed, the square root on the left side is never negative (always nonnegative, see the definition of a radical in Session 6), and by adding it to 7, we cannot get 6.

Example 8.7. Solve the equation $\sqrt{4x^2 + 14x + 3} - 2x - 3 = 0$.

Solution. Here again the radical is not isolated, and we isolate it by moving terms $2x$ and 3 to the right side:

$$\sqrt{4x^2+14x+3}=2x+3.$$

Now square both sides:

$$\left(\sqrt{4x^2+14x+3}\right)^2=(2x+3)^2 \Rightarrow 4x^2+14x+3=4x^2+12x+9.$$

Reducing (cancel out) $4x^2$ from both sides, we get

$$14x+3=12x+9 \Rightarrow 2x=6 \Rightarrow x=3.$$

Finally, we check $x=3$ with the original equation:

$$\sqrt{4\cdot3^2+14\cdot3+3}-2\cdot3-3=\sqrt{36+42+3}-6-3=\sqrt{81}-9=9-9=0.$$

So, $x=3$ is a solution.

Example 8.8. Solve the equation $\sqrt{2x+11}-x=4$.

Solution. Here the radical is not isolated, and we isolate it by moving x to the right side: $\sqrt{2x+11}=4+x$. Now we square both sides:

$$\left(\sqrt{2x+11}\right)^2=(4+x)^2 \Rightarrow 2x+11=16+8x+x^2.$$

We obtained a quadratic equation. Let's write it in the standard form $ax^2+bx+c=0$. For this, we first switch left and right sides: $16+8x+x^2=2x+11$, and then move $2x+11$ from the right side to the left:

$$16+8x+x^2-2x-11=0 \text{ or } x^2+6x+5=0.$$

The last equation can be solved by factoring: $(x+1)(x+5)=0$. We've got two solutions of the quadratic equation: $x=-1$ and $x=-5$. We need to check them with the original equation.

$x=-1$: $\sqrt{2x+11}-x=\sqrt{2\cdot(-1)+11}-(-1)=\sqrt{9}+1=3+1=4$. So, $x=-1$ is a solution.

$x=-5$: $\sqrt{2x+11}-x=\sqrt{2\cdot(-5)+11}-(-5)=\sqrt{1}+5=1+5=6\neq4$. So, $x=-5$ is not a solution and we reject it.

Final answer: the original equation has only one solution $x=-1$.

Example 8.9. Solve the equation $\sqrt{2x-6}+3=x$.

Solution. Here the radical is not isolated, and we isolate it by moving 3 to the right side: $\sqrt{2x-6}=x-3$. Now we square both sides:

$$\left(\sqrt{2x-6}\right)^2=(x-3)^2 \Rightarrow 2x-6=x^2-6x+9 \Rightarrow 0=x^2-6x+9-2x+6 \Rightarrow 0=x^2-8x+15$$
$$\Rightarrow x^2-8x+15=0 \Rightarrow (x-3)(x-5)=0.$$

We've got two solutions of the quadratic equation: $x = 3$ and $x = 5$. We need to check them with the original equation.

$x = 3$: $\sqrt{2x - 6} + 3 = \sqrt{2 \cdot 3 - 6} + 3 = \sqrt{0} + 3 = 3$. So, $x = 3$ is a solution.

$x = 5$: $\sqrt{2x - 6} + 3 = \sqrt{2 \cdot 5 - 6} + 3 = \sqrt{4} + 3 = 2 + 3 = 5$. So, $x = 5$ is a solution.

Final answer: the original equation has two solutions, $x = 3$ and $x = 5$.

Exercises 8

In exercises 8.1 – 8.16, rationalize the denominator and simplify.

8.1. $\dfrac{5}{\sqrt{6}}$

8.2. $\dfrac{4}{\sqrt{3}}$

8.3. $\sqrt{\dfrac{3}{5}}$

8.4. $\sqrt{\dfrac{6}{7}}$

8.5. $\dfrac{2}{3\sqrt{20}}$

8.6. $\dfrac{3}{5\sqrt{27}}$

8.7. $\dfrac{4}{5\sqrt{3}}$

8.8. $\dfrac{2}{3\sqrt{5}}$

8.9. $\dfrac{4}{5+\sqrt{3}}$

8.10. $\dfrac{2}{3+\sqrt{5}}$

8.11. $\dfrac{1}{2-\sqrt{3}}$

8.12. $\dfrac{1}{5+\sqrt{6}}$

8.13. $\dfrac{1}{\sqrt{6}+\sqrt{3}}$

8.14. $\dfrac{1}{\sqrt{6}-\sqrt{5}}$

8.15. $\dfrac{\sqrt{u}+2}{4\sqrt{u}-3\sqrt{v}}$

8.16. $\dfrac{\sqrt{a}+\sqrt{b}}{2\sqrt{a}+5\sqrt{b}}$

In exercises 8.17 – 8.30, solve the given equation.

8.17. $\sqrt{3x+4}=5$

8.18. $\sqrt{4x-3}=7$

8.19. $\sqrt{5x-4}+2=8$

8.20. $\sqrt{7x+1}-4=2$

8.21. $\sqrt{6x-8}+5=3$

8.22. $\sqrt{8x+9}+7=2$

8.23. $\sqrt{16x^2+70x-7}-4x-7=0$

8.24. $\sqrt{25x^2-12x+7}-5x+1=0$

8.25. $\sqrt{9x^2+7x-18}-4=3x$

8.26. $\sqrt{36x^2-19x-25}+3=6x$

8.27. a) $\sqrt{3x+10} - x = 2$

    b) $\sqrt{3x+7} - 3 = x$

8.28. a) $\sqrt{4x+13} - x = 4$

    b) $\sqrt{4x+21} - 4 = x$

8.29. a) $\sqrt{4x-3} - x = 0$

    b) $\sqrt{4x+5} - x = 0$

8.30. a) $\sqrt{6x-8} - x = 0$

    b) $\sqrt{6x+7} - x = 0$

Note. Challenge Problem 10.10 from Ch. 10 represents a general form of equations, as in problems 8.29 and 8.30.

Challenge Problems

8.31. Consider the equation $\sqrt{ax+b^2} - x = b$.
Prove the following statements:

   1) If $b < 0$, then

      a) If $a < b$, then the equation has no solutions.
      b) If $a \geq b$, then the equation has the only solution $x = a - 2b$.

   2) If $b \geq 0$, then

      a) If $a < b$ or $a = 2b$, then the equation has the only solution $x = 0$.
      b) If $a \geq b$ and $a \neq 2b$, then the equation has two solutions
        $x = 0$ and $x = a - 2b$.

8.32. Consider the equation $\sqrt{ax+b^2-ac} - x = b - c$, where $a > 0$.
Prove the following statements:

   1) If $b < 0$, then the equation has only one solution $x = a + c - 2b$.

   2) If $b > a$ or $b = a/2$, then the equation has only one solution $x = c$.

   3) If $0 \leq b \leq a$ and $b \neq a/2$, then the equation has both of the above solutions.

Hints: 1) Check that the quadratic equation $ax + b^2 - ac = (x + b - c)^2$ has the
          solutions $x = c$ and $x = a + c - 2b$.

     2) Use the property $\sqrt{u^2} = |u| = \begin{cases} u, & \text{if } u \geq 0, \\ -u, & \text{if } u < 0. \end{cases}$

Session 9

Complex Numbers and
Squared Form of Quadratic Equations

In section 2 we considered quadratic equations that can be solved by factoring. However, not every quadratic equation can be solved by factoring (using rational numbers). Below we consider corresponding examples. Also, unlike linear equations, quadratic equations do not always have real solutions. We will discussion this case first.

Complex Numbers

A simple example of an equation without real roots is the equation $x^2 + 1 = 0$. It can be written as $x^2 = -1$. Obviously, this equation does not have solutions because the square of a real number cannot be negative.

However, it is possible to introduce a special symbol that can be treated as a solution of the equation $x^2 + 1 = 0$. Usually, this symbol is denoted by the letter i and is called the **imaginary unit** (that's why the letter i). Of course, i is not a real number. It has the property that $i^2 = -1$. Also, we can write that $i = \sqrt{-1}$.

Note. You may be disappointed with such a "definition" of the number i. Indeed, it looks like we introduce an object that does not exist: the equation $x^2 + 1 = 0$ does not have real solutions, and we use the letter i for a non-existent solution. If you have such feelings, you are not the only one. For more than two hundred years many mathematicians felt the same way. Only in the 18th century the exact theory of the so-called **complex numbers** was created which included the symbol i as well as other related to it "magic" numbers. We can look at a complex number as an ordered pair of two real numbers.

Definition. A complex number z is an expression of the form $z = a + bi$. Here a and b are two real numbers, and i is a symbol with the property $i \cdot i = i^2 = -1$. In fact, we are saying that $i \cdot i$ is equal to –1 **by definition** of i. The symbol i is called the **imaginary unit**, the number a is the **real part**, and the number b is the **imaginary part** of the complex number z. The form $a + bi$ is called the **standard form** of a complex number.

Using the symbol i, a square root of any negative number can be written as a complex number: if a is a non-negative real number, then $\sqrt{-a} = \sqrt{a} \cdot \sqrt{-1} = \sqrt{a} \cdot i$, so

$$\boxed{\sqrt{-a} = \sqrt{a} \cdot i} \quad , \; a \geq 0.$$

Example 9.1. Express $\sqrt{-48}$ in terms of i and simplify.

Solution. $\sqrt{-48} = \sqrt{48} \cdot i = \sqrt{16 \cdot 3} \cdot i = 4\sqrt{3} \cdot i$.

We can operate with complex numbers in the same way as with polynomials or rational expressions: we can perform all arithmetic operations with them, distribute, combine like

terms. The only specific property we use is that if we multiply i by i, the result is -1. We say that two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are equal, if separately their real and imaginary parts are equal: $a_1 = a_2$ and $b_1 = b_2$.

If a is a real number, then it can be written as a complex one via: $a = a + 0i$. In this way we can consider the set of real numbers as a subset of the set of complex numbers. In other words, the set of complex numbers can be treated as an extension of the set of real numbers.

Let's consider arithmetic operations with complex numbers.

Example 9.2.

1) Add $(3 + 2i) + (1 + 5i)$.
2) Subtract $(6 + 3i) - (4 - 7i)$.
3) Multiply $(1 + i)(2 + 3i)$.
4) Multiply $(3 + 4i)(3 - 4i)$.

Solution. As we mentioned above, we can perform these operations as with ordinary algebraic expressions just keeping in mind that $i^2 = -1$.

1) $(3 + 2i) + (1 + 5i) = (3 + 1) + (2 + 5)i = 4 + 7i$.

2) $(6 + 3i) - (4 - 7i) = (6 - 4) + (3 - (-7))i = 2 + 10i$.

3) $(1 + i)(2 + 3i) = 2 + 3i + 2i + 3i^2 = 2 + 3i + 2i + 3 \cdot (-1) = 2 + 3i + 2i - 3 = -1 + 5i$.

4) We can proceed the same way as in part 3) or use as a shortcut the difference of squares formula that we already used in session 2 (see the note after example 2.1): $(a - b)(a + b) = a^2 - b^2$. Using this formula we have

$$(3 + 4i)(3 - 4i) = 3^2 - (4i)^2 = 9 - 16 \cdot (-1) = 9 + 16 = 25.$$

Note that the result of multiplication here is a real number. Below we will use a similar result in a general form.

Next, consider division of complex numbers. At first glance, the quotient of two complex numbers does not look like a complex number. For example, can the quotient $\dfrac{5 + 4i}{3 - 2i}$ be represented as a complex number in the standard form $a + bi$? It turns out, the answer is yes. The method of dividing complex numbers is similar to the one we used in the previous session to rationalize the denominator of radical expressions with two terms in the denominator: we multiplied the numerator and denominator of a fraction by the expression conjugate to the denominator. Similarly, consider the following pair of complex numbers z_1 and z_2:

$$z_1 = a + bi, \quad z_2 = a - bi.$$

These numbers have the same real part a, while their imaginary parts b and $-b$ differ only by a sign. Numbers $a + bi$ and $a - bi$ are called **complex conjugate** to each other. An important property is that if we multiply them, the product is a real number:

$$z_1 \cdot z_2 = (a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2.$$

This property allows us to represent the quotient $\dfrac{c+di}{a+bi}$ as a complex number in standard form by multiplying the numerator and denominator of this fraction by $a-bi$ which is the conjugate of the denominator:

$$\frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \frac{ac-bci+adi-bdi^2}{a^2+b^2} = \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}$$

So,

$$\frac{c+di}{a+bi} = \frac{ac+bd}{a^2+b^2} + \frac{ad-bc}{a^2+b^2}i.$$

As you see, the result is written as the sum of two parts, the real part being $\dfrac{ac+bd}{a^2+b^2}$ and the imaginary part being $\dfrac{ad-bc}{a^2+b^2}$. Therefore, the result of division is a complex number in standard form. We got the general formula for dividing two complex numbers.

Note. The above formula looks rather complicated. Don't worry: you don't need to memorize it. Just keep in mind the **method** for dividing complex numbers: multiply the numerator and denominator by the number conjugate to the denominator.

Example 9.3. Divide $(5 + 4i)$ by $(3 - 2i)$.

Solution.

$$\frac{5+4i}{3-2i} = \frac{(5+4i)(3+2i)}{(3-2i)(3+2i)} = \frac{15+10i+12i+8i^2}{9+4} = \frac{(15-8)+(10+12)i}{13} = \frac{7+22i}{13} = \frac{7}{13} + \frac{22}{13}i.$$

If the denominator of a fraction contains only the imaginary part (so, the real part is equal to zero), to divide, simply multiply the numerator and denominator by i.

Example 9.4. Divide 1 by i.

Solution. $\dfrac{1}{i} = \dfrac{1 \cdot i}{i \cdot i} = \dfrac{i}{-1} = -i$. This result can also be written as $i^{-1} = -i$.

Example 9.5. Divide $7 - 5i$ by $4i$.

Solution. $\dfrac{7-5i}{4i} = \dfrac{(7-5i) \cdot i}{4i \cdot i} = \dfrac{7i-5i^2}{4i^2} = \dfrac{7i+5}{-4} = -\dfrac{5}{4} - \dfrac{7}{4}i.$

Now we consider how to raise the imaginary unit i to any positive integer power.

Example 9.6. Calculate

    1) i^3 2) i^4 3) i^5 4) i^6

Solution.

1) $i^3 = i^2 \cdot i = (-1) \cdot i = -i$.   2) $i^4 == \left(i^2\right)^2 = (-1)^2 = 1$.

3) $i^5 = i^4 \cdot i = (i^2)^2 \cdot i = (-1)^2 \cdot i = i$.   4) $i^6 = \left(i^2\right)^3 = (-1)^3 = -1$.

The above example suggests a method for calculating i^n for any positive integer power n. First of all, note that $(-1)^k = 1$, if k is even, and $(-1)^k = -1$, if k is odd. To calculate i^n, consider separately cases of even and odd n.

1) If n is even, it can be written as $n = 2k$, where k is an integer. Then

$$i^n = i^{2k} = \left(i^2\right)^k = (-1)^k.$$

   From here i^n is equal to 1 or -1, depending on whether k is even or odd. For even k, $i^n = 1$, and for odd k, $i^n = -1$.

2) If n is odd, it can be written as $n = 2k + 1$, where k is an integer. Then

$$i^n = i^{2k+1} = i^{2k} \cdot i = \left(i^2\right)^k \cdot i = (-1)^k \cdot i.$$

   From here i^n is equal to i or $-i$, depending on whether k is even or odd. For even k, $i^n = i$, and for odd k, $i^n = -i$.

Example 9.7. Calculate

    1) i^{100} 2) i^{50} 3) i^{29} 4) i^{35}

Solution.

1) $i^{100} = \left(i^2\right)^{50} = (-1)^{50} = 1$.

2) $i^{50} = \left(i^2\right)^{25} = (-1)^{25} = -1$.

3) $i^{29} = i^{28} \cdot i = (i^2)^{14} \cdot i = (-1)^{14} \cdot i = i$.

4) $i^{35} = i^{34} \cdot i = \left(i^2\right)^{17} \cdot i = (-1)^{17} \cdot i = -i$.

The Squared Form of Quadratic Equations

We will call the quadratic equation written in the **squared form**, if it is written as

$$(px + q)^2 = r.$$

In the next session, we will show that any quadratic equation can be transformed to this form.

To solve this equation, we will use the **square-root property**. It states that the equation

$x^2 = r$ has two solutions: $x = \sqrt{r}$ and $x = -\sqrt{r}$. Using this property, we take the square root on both sides of the above equation in the squared form. As a result, we get two linear equations: $px + q = \sqrt{r}$ and $px + q = -\sqrt{r}$. It is very common to write both equations as a single one using the symbol "$\pm$": $px + q = \pm\sqrt{r}$. From here, $px = -q \pm \sqrt{r}$, and $x = \dfrac{-q \pm \sqrt{r}}{p}$. This is the final answer for the solutions of the equation $(px + q)^2 = r$.

Note. Keep in mind that the formula $x = \dfrac{-q \pm \sqrt{r}}{p}$ represents two solutions:

$$x_1 = \frac{-q + \sqrt{r}}{p} \text{ and } x_2 = \frac{-q - \sqrt{r}}{p}.$$

Example 9.8. Solve the quadratic equation $(3x + 2)^2 = 5$.

Solution. Applying the square-root property, take the square root on both sides using the symbol "$\pm$": $3x + 2 = \pm\sqrt{5}$. From here, $3x = -2 \pm \sqrt{5}$ and $x = \dfrac{-2 \pm \sqrt{5}}{3}$.

Note. The final answer $x = \dfrac{-2 \pm \sqrt{5}}{3}$ represents two **exact** solutions written in radical form: $x_1 = \dfrac{-2 + \sqrt{5}}{3}$ and $x_2 = \dfrac{-2 - \sqrt{5}}{3}$. If we want to get solutions as decimal numbers, we can do this only approximately. Using a calculator to approximate $\sqrt{5}$ as 2.236, we can get the following approximations for the solutions:

$$x_1 = \frac{-2 + \sqrt{5}}{3} \approx \frac{-2 + 2.236}{3} \approx 0.079 \text{ and } x_2 = \frac{-2 - \sqrt{5}}{3} \approx \frac{-2 - 2.236}{3} = -1.412$$

Example 9.8 shows us that not every quadratic equation can be solved by factoring in terms of rational numbers.

Example 9.9. Solve the quadratic equation $(5x - 2)^2 = 0$.

Solution. By taking the square root on both sides, we get only one linear equation $5x - 2 = 0$. From here, $x = \dfrac{2}{5}$. So, the given quadratic equation has only one (unique) solution $x = \dfrac{2}{5}$.

Example 9.10. Solve the quadratic equation $x^2 + 5 = 0$.

Solution. Write this equation in squared form $x^2 = -5$. From here, $x = \pm\sqrt{-5}$. Number $\sqrt{-5}$ can be written in terms of the imaginary unit i: $\sqrt{-5} = \sqrt{5} \cdot i$, so $x = \pm\sqrt{5} \cdot i$. The solutions are two imaginary numbers $\sqrt{5} \cdot i$ and $-\sqrt{5} \cdot i$.

Example 9.11. Solve the quadratic equation $(4x-3)^2 + 7 = 0$.

Solution. Write the equation in squared form: $(4x-3)^2 = -7$. Next, apply the square-root property (take square root on both sides) to get $4x-3 = \pm\sqrt{-7}$. Number $\sqrt{-7}$ can be written in terms of imaginary unit i: $\sqrt{-7} = \sqrt{7} \cdot i$ and we get the equation $4x-3 = \pm\sqrt{7} \cdot i$. From here, $4x = 3 \pm \sqrt{7} \cdot i$, and $x = \dfrac{3 \pm \sqrt{7} \cdot i}{4}$. So, the final answer is represented by two complex conjugate numbers:

$$x_1 = \frac{3+\sqrt{7} \cdot i}{4} = \frac{3}{4} + \frac{\sqrt{7}}{4}i \text{ and } x_2 = \frac{3-\sqrt{7} \cdot i}{4} = \frac{3}{4} - \frac{\sqrt{7}}{4}i.$$

Exercises 9

In exercises 9.1 and 9.2, represent the given expression in terms of i and simplify.

9.1.   a) $\sqrt{-25}$

   b) $\sqrt{-32}$

9.2.   a) $\sqrt{-16}$

   b) $\sqrt{-27}$

In exercises 9.3 and 9.4, add or subtract the complex numbers.

9.3.   a) $(7+3i)+(2-5i)$

   b) $(4-6i)-(8+2i)$

9.4.   a) $(9-4i)+(3-i)$

   b) $(6-3i)-(2-8i)$

In exercises 9.5 and 9.6, multiply the complex numbers.

9.5.   a) $3(2-4i)$

   b) $4i(5+2i)$

   c) $(4+5i)(2+3i)$

   d) $(6-3i)(2-8i)$

   e) $(5+3i)(6-5i)$

   f) $(3-2i)(3+2i)$

9.6.   a) $4(3+5i)$

   b) $7i(6-3i)$

   c) $(8+3i)(2+4i)$

   d) $(7-3i)(5-6i)$

   e) $(4-6i)(7+i)$

   f) $(2+4i)(2-4i)$

In exercises 9.7 and 9.8, divide. Write the answer in the standard form for complex numbers.

9.7.   a) $(8+2i)\div 5$

   b) $3\div(2i)$

   c) $(9-6i)\div(-4i)$

   d) $(-4+3i)\div(5-6i)$

   e) $(6+7i)\div(-3+4i)$

   f) $(3-2i)\div(3+2i)$

9.8.   a) $(-6+7i)\div 2$

   b) $2\div(-3i)$

   c) $(4+3i)\div(6i)$

   d) $(5-6i)\div(4-3i)$

   e) $(4+5i)\div(-3+2i)$

   f) $(2-7i)\div(2+7i)$

In exercises 9.9 and 9.10, calculate

9.9.   a) i^{2018}

   b) i^{2019}

   c) i^{2020}

   d) i^{2021}

9.10.   a) i^{2023}

   b) i^{2024}

   c) i^{2025}

   d) i^{2026}

In exercises 9.11 and 9.12, solve the given equations.

9.11. a) $(4x-3)^2 = 6$

  b) $x^2 + 7 = 0$

  c) $(6x+4)^2 + 3 = 0$

9.12. a) $(5x+2)^2 = 3$

  b) $x^2 + 3 = 0$

  c) $(7x-5)^2 + 6 = 0$

Session 10

Completing the Square. The Quadratic Formula

In this session, we show that any quadratic equation can be written in the squared form described in the previous session. In this way, we also get a quadratic formula. It allows us to solve any quadratic equation by simply substituting its coefficients into the formula.

Completing the Square

The procedure for converting a quadratic equation from the standard form $ax^2 + bx + c = 0$ into the squared form $(px + q)^2 = r$ is called the **Completing the Square**. This procedure is based on the following two formulas that we already mentioned in session 2 – the square of the sum and the square of the difference formulas:

$$(x + p)^2 = x^2 + 2px + p^2 \text{ (Square of the Sum Formula).}$$

$$(x - p)^2 = x^2 - 2px + p^2 \text{ (Square of the Difference Formula).}$$

We start with a special case: consider how to complete the square for the equation

$$x^2 + 2px + c = 0 \quad (a = 1, b = 2p).$$

We can rewrite it like this (bring c to the right):

$$x^2 + 2px = -c.$$

Compare the left side of this equation with the square of the sum formula (written from right to left)
$$x^2 + 2px + p^2 = (x + p)^2.$$

We see that the left side of our equation $x^2 + 2px = -c$ is not the square $(x + p)^2$ because the term p^2 is missing. To make up for this deficit (to complete the square), we add p^2 to both sides of our equation:

$$x^2 + 2px = -c,$$

$$x^2 + 2px + p^2 = -c + p^2,$$

$$\left(x + p\right)^2 = -c + p^2.$$

We have completed thus the square and presented the equation in the squared form.

Next, consider the case of reduced equation $x^2 + bx + c = 0$ (the case when $a = 1$). By moving c to the right, we get $x^2 + bx = -c$. To make the left side $x^2 + bx$ look like $x^2 + 2px$, we write the coefficient b as $b = 2\dfrac{b}{2}$, so $p = \dfrac{b}{2}$. Then, to complete the square, we add $p^2 = \left(\dfrac{b}{2}\right)^2$ to both sides of equation: $x^2 + bx = -c$:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2 \Rightarrow \left(x + \frac{b}{2}\right)^2 = -c + \frac{b^2}{4}.$$

We can describe getting the term $\left(\frac{b}{2}\right)^2$ that we add to both sides of the equation $x^2 + bx = -c$, as the following rule.

A rule for constructing a third term to be added to $x^2 + bx$ to complete the square:

> Divide coefficient b by 2 and square it

For the general case of the equation $ax^2 + bx + c = 0$ with arbitrary $a \neq 0$, we can take just one additional step: divide both sides of this equation by the leading coefficient a. Then we get the reduced equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Example 10.1. Solve the quadratic equation $x^2 + 6x + 5 = 0$ by completing the square.

Solution. Bring the last term 5 to the right: $x^2 + 6x = -5$. Next, divide coefficient 6 by 2 and square: $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Add this 9 to both sides of the equation $x^2 + 6x = -5$ to get

$x^2 + 6x + 3^2 = -5 + 9 = 4$. Complete the square: $(x+3)^2 = 4$.

We finish solving the equation by applying the square-root property: $x + 3 = \pm\sqrt{4} = \pm 2$. From here, $x = -3 \pm 2$, and we get two solutions: $x_1 = -3 + 2 = -1$, and $x_2 = -3 - 2 = -5$. Final answer: $x_1 = -1$ and $x_2 = -5$.

Example 10.2. Solve the quadratic equation $x^2 - 5x + 3 = 0$ by completing the square.

Solution. Bring the last term 3 to the right: $x^2 - 5x = -3$. Divide coefficient -5 by 2 and square: $\left(-\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$. Add $\frac{25}{4}$ to both sides of the equation $x^2 - 5x = -3$:

$x^2 - 5x + \left(\frac{5}{2}\right)^2 = -3 + \frac{25}{4}$. Complete the square using the square of the difference formula

$\left(x - \frac{5}{2}\right)^2 = \frac{13}{4}$. From here, applying the square-root property

$$x - \frac{5}{2} = \pm\sqrt{\frac{13}{4}} = \pm\frac{\sqrt{13}}{2}, \quad x = \frac{5}{2} \pm \frac{\sqrt{13}}{2} = \frac{5 \pm \sqrt{13}}{2}.$$

The answer is represented in the radical form that combines two solutions

$$x_1 = \frac{5 + \sqrt{13}}{2} \text{ and } x_2 = \frac{5 - \sqrt{13}}{2}$$

in one formula.

Example 10.3. Solve the quadratic equation $3x^2 + 5x + 2 = 0$ by completing the square.

Solution. Here the leading coefficient is not 1, it is 3. We can use the following steps to solve the given equation.

1) Divide both sides by the leading coefficient 3: $x^2 + \dfrac{5}{3}x + \dfrac{2}{3} = 0$.

2) Bring the last term $\dfrac{2}{3}$ to the right: $x^2 + \dfrac{5}{3}x = -\dfrac{2}{3}$.

3) Divide coefficient $\dfrac{5}{3}$ by 2 and square: $\left(\dfrac{5}{6}\right)^2 = \dfrac{25}{36}$:

4) Add the number $\dfrac{25}{36}$ to both sides of the equation $x^2 + \dfrac{5}{3}x = -\dfrac{2}{3}$:

$$x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = -\frac{2}{3} + \frac{25}{36}.$$

5) Complete the square: $\left(x + \dfrac{5}{6}\right)^2 = \dfrac{1}{36}$.

To get solutions, take square root on both sides: $x + \dfrac{5}{6} = \pm\sqrt{\dfrac{1}{36}} = \pm\dfrac{1}{6}$.

Finally, solve for x: $x = -\dfrac{5}{6} \pm \dfrac{1}{6}$. We come up with two solutions: $x = -\dfrac{5}{6} + \dfrac{1}{6} = -\dfrac{4}{6} = -\dfrac{2}{3}$

and $x = -\dfrac{5}{6} - \dfrac{1}{6} = -\dfrac{6}{6} = -1$. Final answer: $x_1 = -\dfrac{2}{3}$ and $x_2 = -1$.

Note. It would be probably easier to solve the equations in examples 10.1 and 10.3 by factoring. However, we used the method of completing the square to demonstrate its universal character: any quadratic equation can be solved by this method, contrary to the method of factoring that does not work in all cases. In Example 10.2 the solutions are irrational numbers (since they contain $\sqrt{13}$) and cannot be obtained by factoring the quadratic equation using rational numbers.

Quadratic Formula

In order to obtain the quadratic formula, we repeat steps that we used in Example 10.3 for the general equation $ax^2 + bx + c = 0$, $a \neq 0$ written in standard form.

1) Divide both sides by the leading coefficient a to get the reduced equation:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

2) Bring the last term $\dfrac{c}{a}$ to the right side: $x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$.

3) Divide coefficient $\dfrac{b}{a}$ by 2 and square: $\left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2}$

4) Add $\dfrac{b^2}{4a^2}$ to both sides of the equation $x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$:

$$x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 = -\dfrac{c}{a} + \dfrac{b^2}{4a^2} \,.$$

5) Complete the square on the left side:

$$x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 = \left(x + \dfrac{b}{2a}\right)^2 \,.$$

On the right side, we combine fractions by getting the common denominator of $4a^2$:

$$-\dfrac{c}{a} + \dfrac{b^2}{4a^2} = -\dfrac{4ac}{4a^2} + \dfrac{b^2}{4a^2} = \dfrac{-4ac + b^2}{4a^2} = \dfrac{b^2 - 4ac}{4a^2} \,.$$

Now we can write

$$\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2} \,.$$

This is the squared form of the equation $ax^2 + bx + c = 0$. To solve it, we apply the square-root property:

$$x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}} = \pm\dfrac{\sqrt{b^2 - 4ac}}{2a}$$

To solve for x, bring the term $\dfrac{b}{2a}$ to the right side:

$$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \,.$$

We get a formula called the **quadratic formula**

$$\boxed{\; x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \;}$$

This formula gives the solutions of any quadratic equation written in standard form

$$ax^2 + bx + c = 0, \; a \neq 0 \,.$$

The quadratic formula also allows us to get an idea about the "nature" of the solutions for quadratic equations. Let's analyze this. First of all, the quadratic formula represents two

solutions x_1 and x_2:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

written together in one formula, using "$\pm$" operation.

Also notice that the main part of this formula is the expression inside square root: $b^2 - 4ac$. This expression is important and it is given a special name.

Definition. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, and is denoted by the letter D: $D = b^2 - 4ac$.

Using the discriminant, the quadratic formula can be written in a slightly simpler form

$$x = \frac{-b \pm \sqrt{D}}{2a} \quad \text{or} \quad x_1 = \frac{-b + \sqrt{D}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{D}}{2a}.$$

Note. For the reduced quadratic equation $x^2 + bx + c = 0$ (when $a = 1$), $\sqrt{D}$, when $D > 0$, has the following interesting geometric interpretation: it is the distance between roots x_1 and x_2 on number line: $x_1 - x_2 = \frac{-b + \sqrt{D}}{2} - \frac{-b - \sqrt{D}}{2} = \frac{\sqrt{D} + \sqrt{D}}{2} = \sqrt{D}.$

The discriminant is a real number, and it may be positive, negative or zero. Let's see how the sign of the discriminant affects the roots x_1 and x_2.

1) Discriminant is positive: $D > 0$. In this case $\sqrt{D}$ is a positive number and the expressions $-b + \sqrt{D}$ and $-b - \sqrt{D}$ are two different real numbers. Therefore, from the quadratic formula, the quadratic equation has two roots x_1 and x_2 which are real numbers and distinct. Roots x_1 and x_2 may be rational or irrational numbers.

2) Discriminant is zero: $D = 0$. In this case roots $x_1 = \frac{-b}{2a}$ and $x_2 = \frac{-b}{2a}$, so roots coincide and the equation has only one real root $x = -\frac{b}{2a}$.

3) Discriminant is negative: $D < 0$. In this case $\sqrt{D}$ is an imaginary number and the equation has two complex roots x_1 and x_2 which are conjugate to each other.

As you can see, there are only three options regarding the nature of solutions for any quadratic equation: it may have one real solution, two (distinct) real solutions, or two complex conjugate solutions. The sign of the discriminant allows us to distinguish between these three cases.

We now consider some examples of using the quadratic formula. You can use any form, with or without a discriminant. We will use the discriminant form

$$x = \frac{-b \pm \sqrt{D}}{2a}, \quad \text{where } D = b^2 - 4ac.$$

Note. When using the quadratic formula, make sure that the quadratic equation is written in the standard form $ax^2 + bx + c = 0$ (the right side must be zero) to identify coefficients a, b, and c correctly.

Example 10.4. Solve the quadratic equation $3x^2 + 6x = 2$ by using the quadratic formula.

Solution. The equation is not in standard form. To get it in standard form, bring 2 from the right side to the left: $3x^2 + 6x - 2 = 0$. Now identify the coefficients and calculate the discriminant:

$$a = 3, \ b = 6, \ c = -2, \text{ so } D = b^2 - 4ac = 6^2 - 4 \cdot 3 \cdot (-2) = 36 + 24 = 60.$$

The discriminant is positive, so our equation has two distinct real solutions

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-6 \pm \sqrt{60}}{2 \cdot 3} = \frac{-6 \pm \sqrt{4 \cdot 15}}{6} = \frac{-6 \pm 2\sqrt{15}}{6} = \frac{-3 \pm \sqrt{15}}{3}.$$

Both solutions are irrational numbers because they contain $\sqrt{15}$.

Example 10.5. Solve the quadratic equation $4x^2 - 20x + 25 = 0$ by using the quadratic formula.

Solution. The equation is already in standard form. We have

$$a = 4, \ b = -20, \ c = 25, \text{ so } D = b^2 - 4ac = (-20)^2 - 4 \cdot 4 \cdot 25 = 400 - 400 = 0.$$

The discriminant is zero, so our equation has only one real solution

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-20) \pm \sqrt{0}}{2 \cdot 4} = \frac{20}{8} = \frac{5}{2}.$$

Example 10.6. Solve the quadratic equation $6x^2 - 5x + 2 = 0$ by using the quadratic formula.

Solution. The equation is already in standard form. We have

$$a = 6, \ b = -5, \ c = 2, \text{ so } D = b^2 - 4ac = (-5)^2 - 4 \cdot 6 \cdot 2 = 25 - 48 = -23.$$

The discriminant is negative, so our equation has two complex conjugate solutions

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-5) \pm \sqrt{-23}}{2 \cdot 6} = \frac{5 \pm \sqrt{23} \cdot i}{12} = \frac{5}{12} \pm \frac{\sqrt{23}}{12} i.$$

Exercises 10

In exercises 10.1 and 10.2, fill in blanks to complete the squares.

10.1. a) $x^2 + 8x + \underline{} = (x + \underline{})^2$

b) $x^2 - 3x + \underline{} = (x - \underline{})^2$

c) $x^2 + x + \underline{} = (x + \underline{})^2$

d) $x^2 - \dfrac{5}{4}x + \underline{} = (x - \underline{})^2$

10.2. a) $x^2 + 6x + \underline{} = (x + \underline{})^2$

b) $x^2 - 9x + \underline{} = (x - \underline{})^2$

c) $x^2 - x + \underline{} = (x - \underline{})^2$

d) $x^2 + \dfrac{7}{3}x + \underline{} = (x + \underline{})^2$

In exercises 10.3 and 10.4, solve the given quadratic equations by **completing the square**.

10.3. a) $x^2 - 4x - 5 = 0$

b) $x^2 + 5x + 6 = 0$

c) $x^2 - 7x + 8 = 0$

d) $x^2 + 8x + 21 = 0$

e) $2x^2 + 3x - 9 = 0$

10.4. a) $x^2 + 2x - 8 = 0$

b) $x^2 - 9x + 14 = 0$

c) $x^2 + 3x - 5 = 0$

d) $x^2 - 10x + 28 = 0$

e) $3x^2 + 7x + 2 = 0$

In exercises 10.5 and 10.6, **without solving** the equations, determine the number of real roots.

10.5. a) $7x^2 - 3x + 2 = 0$

b) $6x^2 + 9x - 4 = 0$

c) $3x^2 - 6x + 3 = 0$

10.6. a) $4x^2 + 24x + 36 = 0$

b) $5x^2 - 6x + 3 = 0$

c) $8x^2 + 5x - 2 = 0$

In exercises 10.7 and 10.8, solve the given quadratic equations using the **quadratic formula**.

10.7. a) $5x^2 - 8x = 1$

b) $16x^2 + 24x + 9 = 0$

c) $4x^2 - 7x + 5 = 0$

d) $-6x^2 + 17x - 12 = 0$

10.8. a) $7x^2 - 6x = 3$

b) $9x^2 - 42x + 49 = 0$

c) $6x^2 + 5x + 3 = 0$

d) $-12x^2 + 28x - 15 = 0$

Challenge Problems

10.9. Construct a quadratic equation with real coefficients having the root $2 + 3i$. How many such equations can you construct?

10.10. Consider the equation $\sqrt{2ax + b} - x = 0$, where a and b are real numbers. Here, we will take into account only real solutions. Prove the following statements:

1) If $b > 0$, then the equation has one solution.

2) If $b = 0$, then

    2.1) If $a \leq 0$, then the equation has one solution.

    2.2) If $a > 0$, then the equation has two solutions.

3) If $b < 0$ and $a \leq 0$, then the equation has no solutions.

4) If $b < 0$ and $a > 0$, then

    4.1) If $b < -a^2$, then the equation has no solutions.

    4.2) If $b = -a^2$, then the equation has one solution.

    4.3) If $b > -a^2$, then the equation has two solutions.

Find solutions in all cases (where the equation has solutions).

10.11. Find the mistake in the following "Theorem".

Theorem: All numbers are equal.

Proof: Let a and b be two numbers. Denote $a + b = u$.

1) Multiply both sides of the above equation by $a - b$:
$$(a + b)(a - b) = u(a - b).$$

2) Apply the difference of squares formula to the left side and distribute the right side:
$$a^2 - b^2 = ua - ub.$$

3) Rearrange the terms:
$$a^2 - ua = b^2 - ub.$$

4) Complete the square on both sides by adding $u^2/4$:
$$a^2 - ua + \frac{u^2}{4} = b^2 - ub + \frac{u^2}{4} \;\Rightarrow\; \left(a - \frac{u}{2}\right)^2 = \left(b - \frac{u}{2}\right)^2.$$

5) Take square root from both sides:
$$a - \frac{u}{2} = b - \frac{u}{2}.$$

6) Cancel out $u/2$, then $a = b$.

Session 11

Parabolas

In this session we will relate the quadratic **equation** $ax^2 + bx + c = 0$ to the quadratic **function** $y = ax^2 + bx + c$. In other words, we will focus not on solving the above equation, but rather on the relationship between x and y. The graph of a quadratic function (and the function itself) is called a **parabola**. Recall that for a quadratic equation there are three cases about the number of real solutions: it may have one solution, two solutions, or no solutions at all. As a graph, the parabola allows us to visualize all three cases, as well as some other properties of the quadratic function.

Let's start with the simplest (or basic) parabola $y = x^2$. Notice, first of all, that this function takes the same values for x and $-x$ since $(-x)^2 = x^2$. Both points $\left(x,\ x^2\right)$ and $\left(-x,\ x^2\right)$ lie on the graph of parabola and they are symmetrical to each other over the y-axis. Therefore, if we draw this parabola only for positive x, then we can reflect this graph over the y-axis to get the entire picture.

Another simple property is that for positive x, the bigger x, the bigger $y = x^2$. We say that parabola $y = x^2$ increases (for positive x). However, this function is **not linear**: its graph is not a straight line. Instead, the graph is a **curve**. To picture this curve, we can calculate several values of the parabola for some values of x. The following table represents one of the possible calculations.

x	0	1	2	3
$y = x^2$	0	1	4	9
(x, y)	(0, 0)	(1, 1)	(2, 4)	(3, 9)

If we plot points (x, y) and connect them with a smooth curve, we will get the picture:
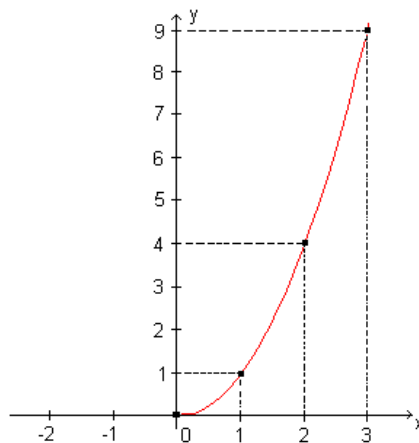
Graph of the parabola $y = x^2$ for nonnegative x.

To get the entire parabola (to include negative x), we reflect this graph over the y-axis. Here is the final picture.

$$y = x^2$$
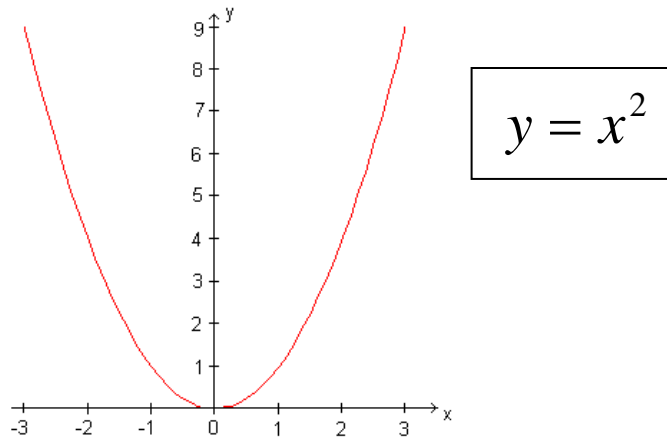
Let's observe this graph. Notice that for negative x parabola is going down from left to right (we say that parabola decreases), and for positive x parabola is going up (increases). We say that this parabola opens up (or upward). Also, it has the lowest point at the origin (0, 0). This point is called the **vertex** of parabola.

Now consider the second basic parabola $y = -x^2$. We do not need any special analysis to graph this function. Point $\left(x,\ x^2 \right)$ lies on the graph of parabola $y = x^2$, and point $\left(x,\ -x^2 \right)$ lies on the graph of parabola $y = -x^2$. These two points are symmetrical to each other over the x-axis. Therefore, to get the graph of $y = -x^2$, we can simply reflect the graph of $y = x^2$ over the x-axis (flip the graph upside-down). We say that parabola $y = -x^2$ opens down (downward). Here is its graph:

$$y = -x^2$$
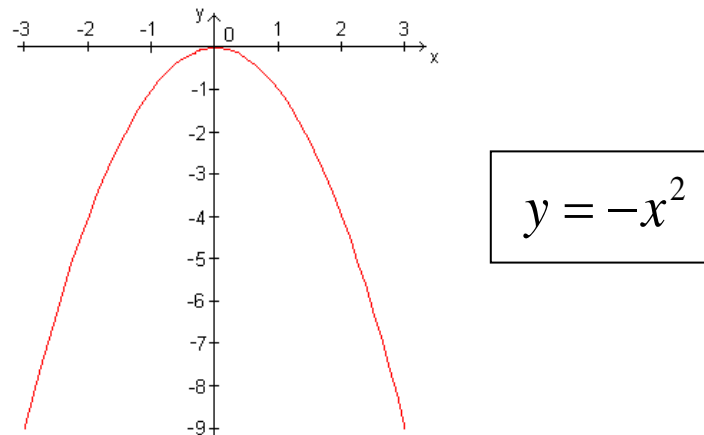
Now consider the general quadratic function $y = ax^2 + bx + c$. It turns out that the shape of its graph is similar to one of the above graphs of $y = x^2$ and $y = -x^2$ (depending on whether the leading coefficient a is positive or negative). To understand why, let's consider three types of transformations (deformations) of graphs of the functions. Namely, assume

that we know the graph of $y = f(x)$, and consider how we can construct graphs of functions $y = af(x)$, $y = f(x) + k$, and $y = f(x+h)$.

1) Compare the following graphs of $y = x^2$, $y = 2x^2$ and $y = \dfrac{1}{2}x^2$.
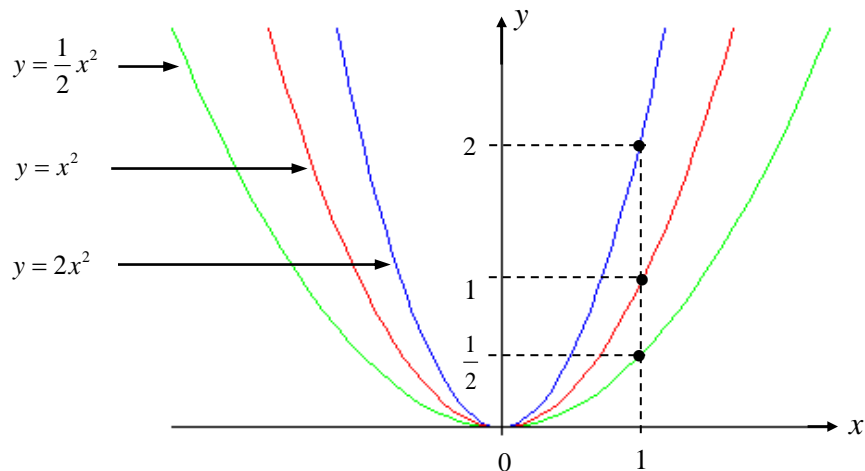
As you see, the vertices of all three parabolas remain at the origin and graphs of $y = 2x^2$ and $y = \dfrac{1}{2}x^2$ resemble the parabola $y = x^2$. To get the graph of $y = 2x^2$, we stretch the graph of $y = x^2$ in the y-axis, and to get the graph of $y = \dfrac{1}{2}x^2$ we compress the graph of $y = x^2$ in the y-axis. More generally: the graph of $y = ax^2$ is obtained from the graph of $y = x^2$ by stretching along the y-axis if $a > 1$ (the graph gets narrower), and by compressing along the y-axis if $0 < a < 1$ (the graph gets wider). The graph of $y = -ax^2$ is obtained from the graph of $y = ax^2$ by reflection over the x-axis.

For an arbitrary function $y = f(x)$, we can get the graph of $y = af(x)$ for positive a in a similar way: we stretch or compress the graphs of $y = f(x)$ along the y-axis depending on whether $a > 1$ or $a < 1$. So, the shape of the graph of $af(x)$ resembles the graph of $f(x)$. The graph of $y = -af(x)$ is obtained from the graph of $y = af(x)$ by reflection over the x-axis and resembles the graph of $y = -f(x)$.

2) The graph of $y = f(x) + k$ is obtained from the graph $y = f(x)$ by its shifting along the y-axis by k units. If $k > 0$, the graph is shifted up, and if $k < 0$ – down. So, the transformation $f(x) \to f(x) + k$ does not change the shape of the graph of $f(x)$, it only changes the **position** of the graph.

3) Finally, consider the graph of $y = f(x+h)$. This graph is obtained from the graph of $y = f(x)$ by its horizontal shifting along the x-axis by h units. It is important not to mix up the direction of shifting: to the left or to the right. It may seem that for positive h the graph is shifted to the right, and for negative h – to the left. However, this is wrong. The correct answer is just the opposite: **if $h > 0$ (h is positive), graph is shifted**

to the left, and if $h < 0$ (h is negative), to the right. Here is the reason. Let's denote $g(x) = f(x + h)$. For positive h, consider two points x_0 and $x_1 = x_0 - h$. Point x_1 lies to the left of x_0. At point x_1, function $g(x)$ takes the value

$$g(x_1) = f(x_1 + h) = f(x_0 - h + h) = f(x_0),$$

which is the same as the value of $f(x)$ at point x_0. Since $x_1 < x_0$, and $g(x_1) = f(x_0)$, we have a shift to the left. A similar reasoning is true when h is negative.

In the case of a parabola, we consider the function $y = (x-h)^2$. Similar to the above arbitrary function $f(x)$, we conclude that the shape of the parabola $y = (x-h)^2$ is exactly the same as the shape of $y = x^2$, and only the location is different: if $h > 0$, $y = (x-h)^2$ is located h units to the right of $y = x^2$, and if $h < 0$, h units to the left. The same is true for functions $y = -(x-h)^2$ and $y = -x^2$. The vertex of both parabolas $y = (x-h)^2$ and $y = -(x-h)^2$ has the coordinates $(h, 0)$.

If we combine together the above three transformations, we see that if a quadratic function is written in the **squared form** $y = a(x-h)^2 + k$, then its graph resembles the basic parabolas $y = x^2$ or $y = -x^2$. In particular, the vertex of the parabola $y = a(x-h)^2 + k$ (i.e. its lowest or highest point) has the coordinates (h, k). The graph of this parabola can be obtained from the graph of $y = ax^2$ by shifting in the vertical direction by $|k|$ units (up or down) and in the horizontal direction by $|h|$ units (left or right) as described above. The graph is symmetric over the vertical line that passes through its vertex (h, k), so the vertical line $x = h$ is the **line of symmetry** of the parabola. The parabola opens up if $a > 0$ and opens down if $a < 0$.

To graph a parabola that is written in the squared form $y = a(x-h)^2 + k$,

Use the following steps:

1) Plot the vertex (h, k).

2) Draw dotted vertical line through the vertex. This is the line of symmetry of the parabola. Its equation is $x = h$.

3) If you want to find y-intercept, put $x = 0$ and calculate y.

4) If you want to find x-intercepts, solve the equation $a(x - h)^2 + k = 0$.

5) Determine whether the parabola opens up or down depending on the sign of the leading coefficient a: if $a > 0$, it opens up, if $a < 0$, it opens down.

6) Draw the parabola. To be more accurate, you may calculate several values of the parabola and plot corresponding points.

Example 11.1. Graph the parabola $y = 2(x+3)^2 + 4$.

Solution. Let's follow the above steps. We have $a = 2$, $h = -3$, $k = 4$.

1) Plot the vertex $(h, k) = (-3, 4)$:
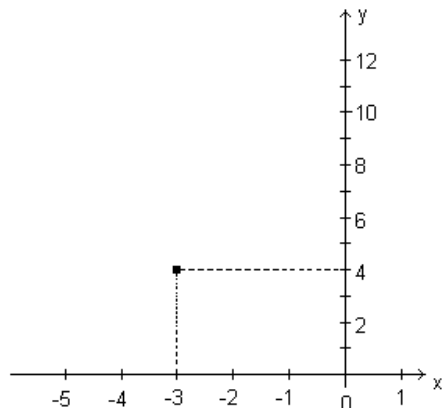
2) Draw dotted vertical line of symmetry through the vertex $(-3, 4)$:
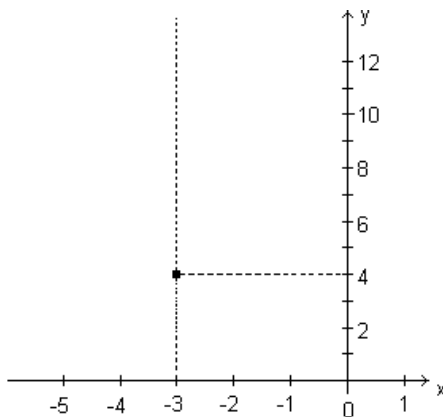
3) Determine how the parabola opens (up or down) by looking at the leading coefficient $a = 2$. It is positive, so parabola opens up.
4) To draw the parabola more accurately, calculate several values (you may choose any values of x, we choose -1 and -2):

$$y(-1) = 2(-1+3)^2 + 4 = 12, \text{ so the graph contains the point } (-1, 12).$$
$$y(-2) = 2(-2+3)^2 + 4 = 6, \text{ so the graph contains the point } (-2, 6).$$

The parabola looks like this
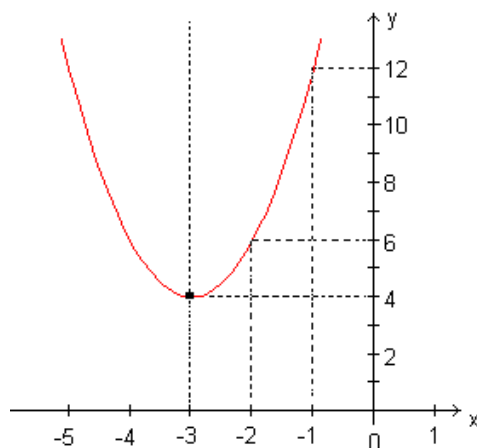
Note. It is not needed to always show the dotted line of symmetry. Final picture may look like this:

$$y = 2(x+3)^2 + 4$$
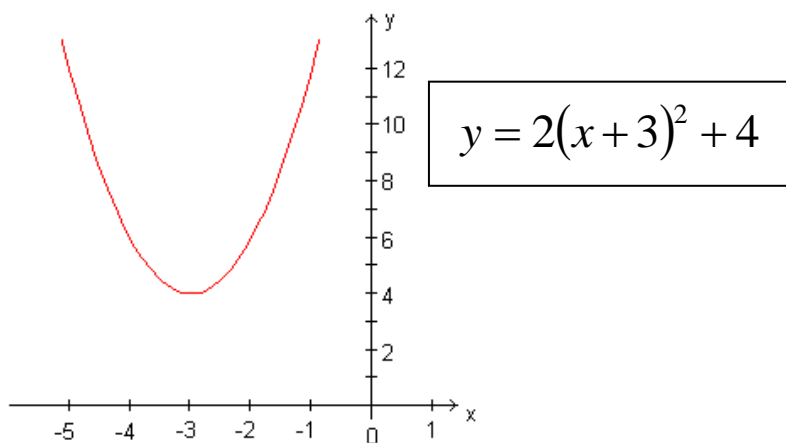
This graph may also be obtained from the graph of $y = 2x^2$ by shifting three units to the left and four units up.

Notice that any quadratic function $y = ax^2 + bx + c$ can be represented in the squared form $y = a(x-h)^2 + k$ (and, therefore, its graph has the same shape as above). Actually, we already did this in the previous session, when we discussed the method of the completing the square for a quadratic equation.

Let's repeat this procedure one more time (with a small modification). We can use these

Steps to transform quadratic function
from the general form to the squared from

1) Divide all term of the equation $y = ax^2 + bx + c$ by a:

$$\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

2) Isolate terms $x^2 + \dfrac{b}{a}x$ on the right side (bring the term $\dfrac{c}{a}$ to the left side):

$$\frac{y}{a} - \frac{c}{a} = x^2 + \frac{b}{a}x$$

3) Complete the square on the right side using the procedure described in the previous session:

a) Divide coefficient $\dfrac{b}{a}$ by 2 and square it: $\dfrac{b}{a} \div 2 = \dfrac{b}{2a}$, $\left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2}$.

b) Add the term $\dfrac{b^2}{4a^2}$ to both sides of the equation in part 2):

$$\frac{y}{a} - \frac{c}{a} + \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

c) Complete the square:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{a}\right)^2$$

4) On the left side of the equation, combine terms $-\dfrac{c}{a}$ and $\dfrac{b^2}{4a^2}$:

$$-\frac{c}{a} + \frac{b^2}{4a^2} = -\frac{4ac - b^2}{4a^2}$$

Equation takes the form

$$\frac{y}{a} - \frac{4ac - b^2}{4a^2} = \left(x + \frac{b}{a}\right)^2$$

5) Solve this equation for y:

a) $\dfrac{y}{a} = \left(x + \dfrac{b}{a}\right)^2 + \dfrac{4ac - b^2}{4a^2}$

b) $y = a\left(x + \dfrac{b}{a}\right)^2 + \dfrac{4ac - b^2}{4a}$

We have written the parabola $y = ax^2 + bx + c$ in the squared form $y = a(x - h)^2 + k$, where

$$h = -\frac{b}{2a}, \quad k = \frac{4ac - b^2}{4a}.$$

Actually, to draw the graph of the parabola $y = ax^2 + bx + c$, you do not need to go through the above steps every time. Just memorize the most important formula for h, which is **the x-coordinate of the vertex of parabola**. We will denote it by x_v:

$$x_v = -\frac{b}{2a}$$

This is the first coordinate for the vertex (h, k). The second coordinate k we will denote by y_v:

$$y_v = \frac{4ac - b^2}{2a}$$

This formula is more complicated, and you do not need to memorize it. The coordinate y_v can be calculated by substitution the value of x_v for x in the original parabola $y = ax^2 + bx + c$. Here are possible

Steps to graph the general parabola $y = ax^2 + bx + c$

1) Identify coefficients a, b, and c.

2) Calculate the x-coordinate x_v of the vertex of the parabola by the formula: $x_v = -\dfrac{b}{2a}$
   or by the competing the square.

3) Calculate the y-coordinate y_v of the vertex by substituting x_v in the original equation.

4) Follow the above steps for graphing the parabola $y = a(x - h)^2 + k$, where
   $$h = x_v, \ \ k = y_v.$$

Example 11.2. Graph the parabola $y = -2x^2 + 8x - 5$.

Solution.

1) Identify the coefficients a, b, and c: $a = -2$, $b = 8$, $c = -5$.

2) Calculate the x-coordinate of the vertex: $x_v = -\dfrac{b}{2a} = -\dfrac{8}{2 \cdot (-2)} = 2$.

3) Calculate the y-coordinate of the vertex by substitution $x_v = 2$ in the original
   equation: $y_v = -2 \cdot 2^2 + 8 \cdot 2 - 5 = 3$.
   So, the vertex of the parabola has the coordinates (2, 3).

4) Draw the parabola according to the steps described for the squared form
   $y = a(x - h)^2 + k$. In particular, parabola opens down (because $a = -2 < 0$) and has a
   vertical line of symmetry that passes through the vertex (2, 3). Also (for more
   accuracy), we can calculate the values $y(0) = -5$, and $y(1) = 1$. Here is the picture.
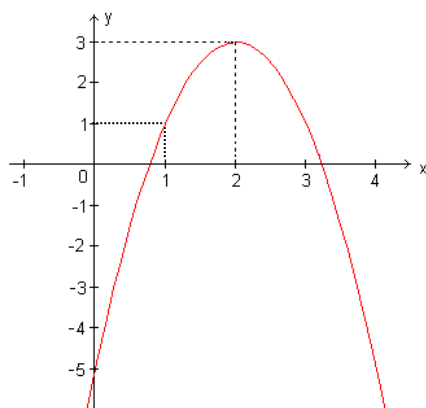
$$y = -2x^2 + 8x - 5$$

Note. In order to plot the parabola more accurately, you may want to find the x- and y-intercepts. It is easy to find the y-intercept. Indeed, any point on the y-axis has the first coordinate zero, so just substitute $x = 0$ into the function $y = ax^2 + bx + c$, and you will find that the y-intercept is equal to c. To find x-intercepts, substitute $y = 0$ and solve the quadratic equation $ax^2 + bx + c = 0$.

Consider in more details how parabola shows possible cases about the number of real solutions (roots) to the equation $ax^2 + bx + c = 0$. In general, solving the equation $f(x) = 0$ means to find all values of x for which the function $y = f(x)$ takes the value of zero: $y = 0$. Geometrically, points $(x, 0)$ lie on the x-axis. Therefore, roots of the equation $f(x) = 0$ are x-coordinates of points of intersection of the graph of $y = f(x)$ with the x-axis. Thus, geometrically solving the equation $f(x) = 0$ is equivalent to finding all x-intercepts of the graph of the function $y = f(x)$.

In particular, to find real solutions to the quadratic equation $ax^2 + bx + c = 0$, we need to find all x-intercepts of the parabola $y = ax^2 + bx + c$. We consider the case $a > 0$ (parabola opens up). The case $a < 0$ is similar. Obviously, there are only three possible positions of the parabola with respect to x-axis:

1) The vertex of the parabola is located below x-axis. In this case there are two x-intercepts, so there are two roots of the quadratic equation.

2) The parabola touches the x-axis at one point (at the vertex), so there is only one root.

3) The parabola is located above x-axis. In this case, no x-intercepts, so no real roots.

Here are the corresponding pictures:
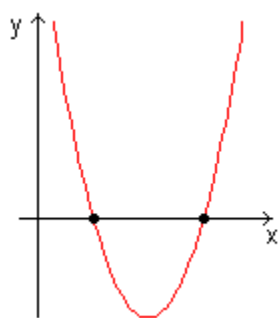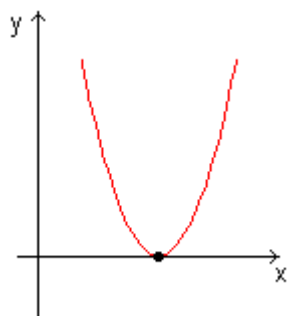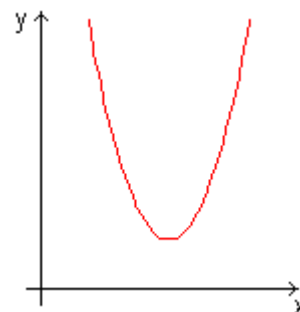
| Two roots | One root | No roots |

For example, parabola $y = x^2 - 4$ has two roots 2 and -2 (since $y\,(2) = y\,(-2\,) = 0$), parabola $y = x^2$ has only one root 0, and parabola $y = x^2 + 4$ does not have roots at all (since the equation $x^2 + 4 = 0$ does not have real solutions).

Exercises 11

In exercises 11.1 and 11.2,

    1) Identify the coordinates of the vertex of given parabola.

    2) Determine whether the parabola opens up or down.

11.1. a) $y = 3(x+4)^2 - 5$ 11.2. a) $y = -5(x-3)^2 + 7$

     b) $y = -4(x-2)^2 + 6$ b) $y = 6(x+7)^2 - 3$

     c) $y = 7(x-5)^2 - 4$ c) $y = -2(x+1)^2 + 8$

In exercises 11.3 and 11.4, the given graphs are shifted graphs of the parabola $y = 3x^2$. Write the equations for the parabolas graphed below,

    1) In squared form $y = a(x-h)^2 + k$.

    2) In general form $y = ax^2 + bx + c$.

11.3.
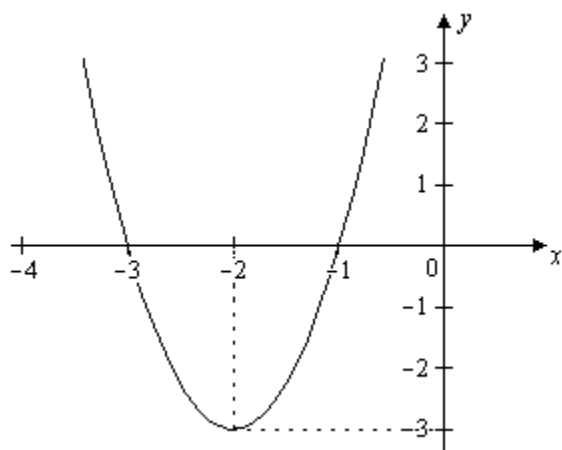
       a)   b)
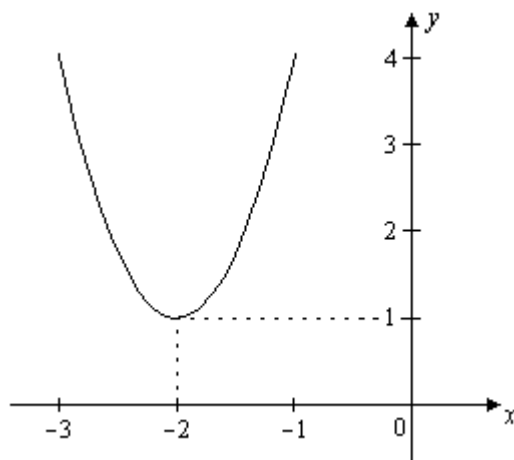
11.4.

a)
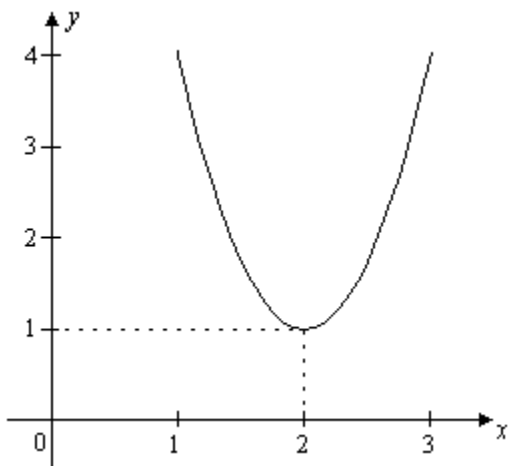
b)
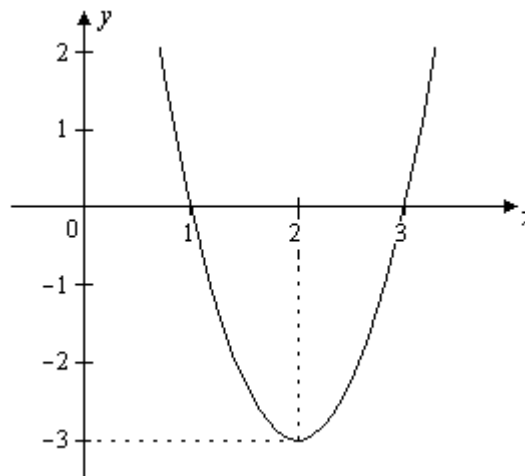
In exercises 11.5 and 11.6, the given graphs are shifted graphs of the parabola $y = -2x^2$.
Write the equations for the parabolas graphed below,

1) In squared form $y = a(x-h)^2 + k$.

2) In general form $y = ax^2 + bx + c$.

11.5.

a)

b)

11.6.

a)

b)
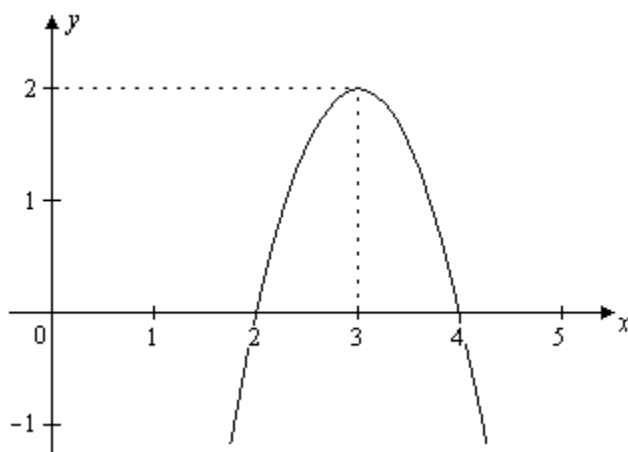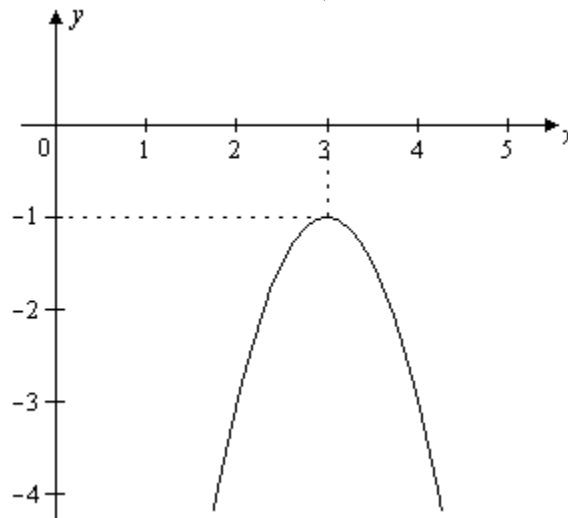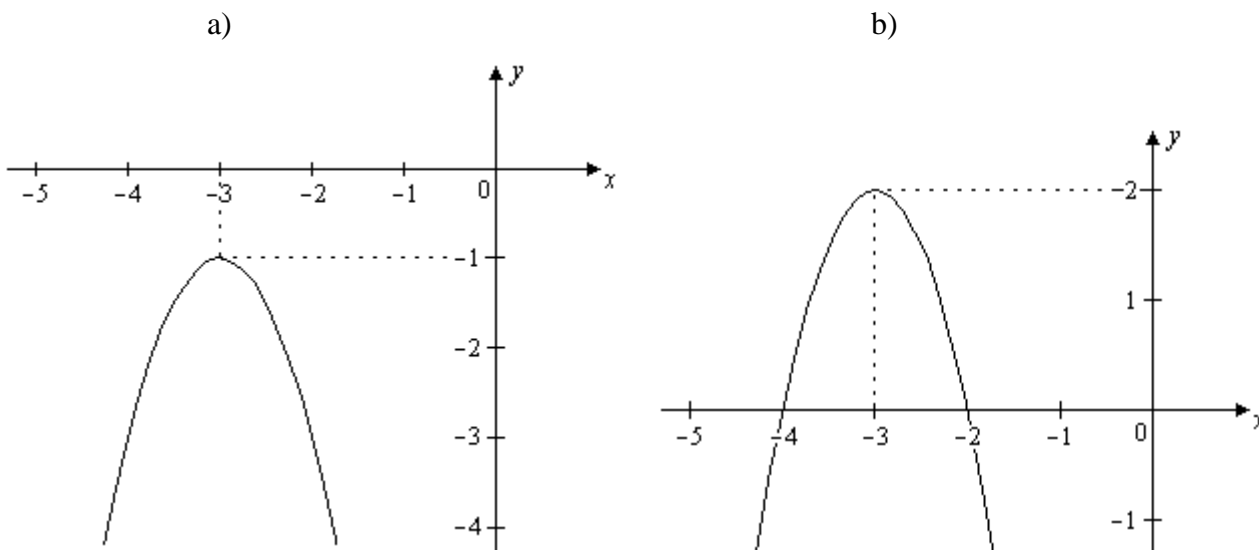
In exercises 11.7 and 11.8, for each of the quadratic functions

    1) Find the vertex of the parabola with the coordinates.

    2) Find the y- intercept.

    3) Find the x- intercepts if they exist.

    4) Write the equation of the line of symmetry.

    5) Graph the parabola.

    6) Label the vertex and indicate its the x- and y- coordinates.

11.7. a) $y = x^2 - 2x - 3$ 11.8. a) $y = -x^2 - 2x + 8$

     b) $y = -x^2 - 4x + 5$ b) $y = x^2 - 6x + 5$

Challenge Problems

11.9. Let the x-intercepts of a parabola be (2, 0) and (4, 0). Find the equation of the line of symmetry of the parabola.

11.10. Let the equation of the parabola's line of symmetry be $x = 5$, and one of the x-intercepts is (3, 0). Find another x-intercept.

Session 12

Distance Formula, Midpoint Formula, and Circles

When we say that a point in the *xy*-plane is given we mean that coordinates of the point are given. If *A* is a point in the plane and (*x*, *y*) are its coordinates in the coordinate system, then we will also denote this point by $A(x, y)$.

Distance Formula

Assume two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given. Consider the problem to find the distance between them (i.e. to get a formula for this distance).

Recall that if we plot a point $C(x_0, y_0)$ in the coordinate system, then x_0 is the horizontal coordinate (along the *x*-axis), and y_0 is the vertical coordinate (along the *y*-axis):
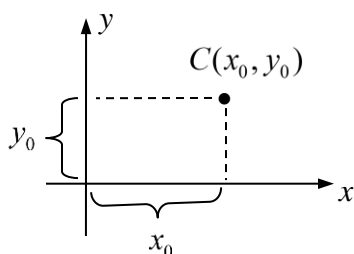
Now consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
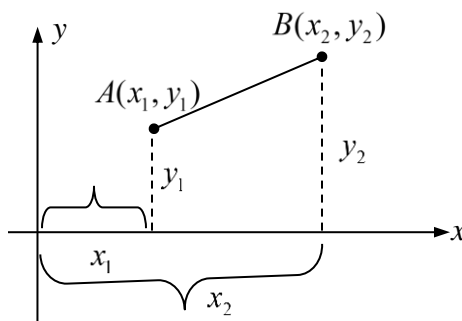
We denote by $d(A, B)$ the distance between points *A* and *B* (i.e. the length of the line segment *AB*). Also, we will use the notation $|AB|$, so $d(A, B) = |AB|$. To find this distance, we draw a right triangle with the hypotenuse *AB* and horizontal and vertical legs *AC* and *BC*:
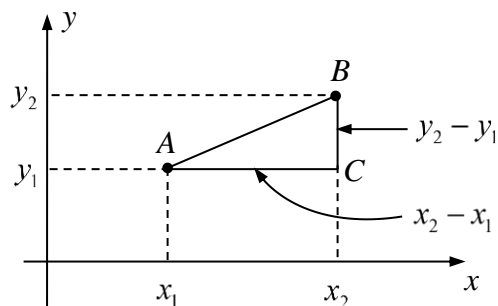
We can see that $|AC| = x_2 - x_1$ and $|BC| = y_2 - y_1$. By the Pythagorean Theorem,

$$d^2(A,B) = |AB|^2 = |AC|^2 + |BC|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 .$$

By taking square root of both sides, we get the **Distance Formula**:

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Here (x_1, y_1) and (x_2, y_2) are coordinates of the points A and B respectively.

Note. The distance formula will not change, if we switch (exchange) x_1 and x_2, and/or y_1 and y_2. The reason is that if a and b are two numbers, then $(a-b)^2 = (b-a)^2$. Also, the distance formula is valid for points located in any quadrant (not only in the 1st quadrant).

Example 12.1. Calculate the distance between points $A(2, -3)$ and $B(-5, -7)$.

Solution. By the distance formula we get:

$$|AB| = \sqrt{(2-(-5))^2 + (-3-(-7))^2} = \sqrt{(2+5)^2 + (-3+7)^2} = \sqrt{7^2 + 4^2} = \sqrt{49+16} = \sqrt{65} .$$

Later, in Session 18 "Solving Oblique Triangles – Law of Cosines", in Example 18.4 we will justify a method for checking whether a triangle with given sides a, b, and c is acute, obtuse or a right triangle. Here are the pictures of such triangles:

Acute triangle   Right triangle   Obtuse triangle
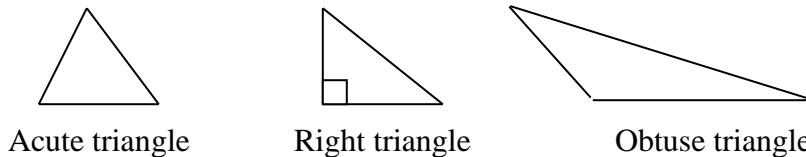
The method is the following:

Let c be the biggest side of the triangle. Calculate the value $E = a^2 + b^2 - c^2$.

    If $E > 0$, then the triangle is acute.

    If $E < 0$, then the triangle is obtuse.

    If $E = 0$, then the triangle is right.

Example 12.2. Consider a triangle ABC with vertices $A(2, -3)$, $B(-5, -7)$ and $C(-2, 6)$. Determine what kind of triangle it is: acute, obtuse, or a right triangle. Also, determine which angle in the triangle ABC is the biggest angle, and which angle is the smallest one.

Solution. First, we calculate squares of all three sides of triangle ABC (there is no need to calculate the sides themselves because the above expression for E contains squares of sides only). The length of the side AB we have already calculated in Example 12.1, so $|AB|^2 = 65$. Using the distance formula for the sides AC and BC we get:

$$|AC|^2 = (2-(-2))^2 + (-3-6)^2 = (2+2)^2 + (-9)^2 = 16+81 = 97 ,$$

$$|BC|^2 = \left(-5 - (-2)\right)^2 + \left(-7 - 6\right)^2 = \left(-5 + 2\right)^2 + \left(-13\right)^2 = 9 + 169 = 178 .$$

We see that side BC is the longest one. Now we can construct the above expression for E:

$$E = |AB|^2 + |AC|^2 - |BC|^2 = 65 + 97 - 178 = -16 .$$

Since E is negative, we conclude that the triangle ABC is obtuse. In any triangle, the larger the side, the greater the opposite angle. In our case, the largest side is BC, and the smallest side is AB. Therefore, the largest angle is at A, and the smallest one is at C.

Midpoint Formula

Let A and B be two points. The midpoint formula gives the coordinates of the point C located in the middle of the line segment AB. To get these coordinates, consider first the simplest case when points A and B lie on the number line (i.e. on the horizontal x-axis):
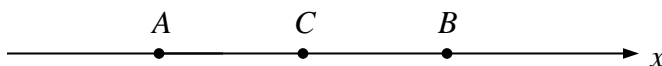
Here point C is the midpoint of the segment AB. The location of any point on the number line is determined by its coordinate (a number). Let a, b and c be the coordinates of points A, B and C respectively. The distance between the points A and B (i.e. the length of the segment AB) is $|AB| = b - a$. It is easy to show that the coordinate c of the midpoint C is the average of coordinates a and b: $c = \dfrac{a+b}{2}$. Indeed, with this coordinate, the distance $|AC|$ is half of the distance $|AB|$:

$$|AC| = c - a = \frac{a+b}{2} - a = \frac{a+b-2a}{2} = \frac{b-a}{2} = \frac{|AB|}{2} .$$

Now, let the points A, B and the midpoint C lie in the plane and have coordinates (x_1, y_1), (x_2, y_2) and (x_m, y_m) respectively:
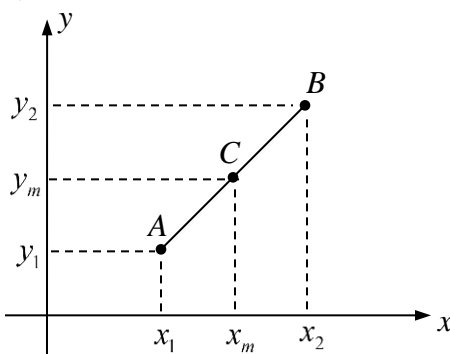
We see that x_m is the midpoint of the segment $[x_1, x_2]$ on the x-axis, and y_m is the midpoint of the segment $[y_1, y_2]$ on the y-axis. Therefore, x_m is the average of x_1 and x_2, and y_m is the average of y_1 and y_2. We thus obtained the **Midpoint Formula**:

$$x_m = \frac{x_1 + x_2}{2}, \ y_m = \frac{y_1 + y_2}{2}$$

In words, coordinates (x_m, y_m) of the midpoint C of the line segment AB are averages of the corresponding coordinates of the endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$.

Example 12.3. Calculate the coordinates of the midpoint of the line segment with endpoints $(-3, 4)$ and $(-7, 10)$.

Solution. Let (x_m, y_m) be coordinates of the midpoint. By the midpoint formula

$$x_m = \frac{-3 + (-7)}{2} = \frac{-10}{2} = -5 \ \text{ and } \ y_m = \frac{4 + 10}{2} = \frac{14}{2} = 7 \ .$$

Answer: the midpoint has coordinates $(-5, 7)$.

Example 12.4. Let $A(4, -7)$ be the endpoint of a line segment, and $C(-6, 9)$ be its midpoint. Find the coordinates of the other endpoint of the line segment.

Solution. Denote the other endpoint by $B(x, y)$. We will use the midpoint formula with the given midpoint $C(-6, 9)$, so $x_m = -6$ and $y_m = 9$. We have

$$x_m = -6 = \frac{4 + x}{2} \ \Rightarrow \ -12 = 4 + x \Rightarrow x = -16,$$

$$y_m = 9 = \frac{-7 + y}{2} \ \Rightarrow \ 18 = -7 + y \Rightarrow y = 25.$$

Answer: endpoint B has coordinates $(-16, 25)$.

Circle

By definition, a circle is a set of points in the plane equidistant (having the same distance) to a fixed point in the plane. This fixed point is called the **center** of the circle, and the distance from any point on the circle to the center is called the **radius**.

Equation of a circle can be easily derived directly from the distance formula. Let $C(a, b)$ be a center of a circle, and $A(x, y)$ be any point on the circle. If r is the radius of the circle, then, by definition, $d(A, C) = r$.
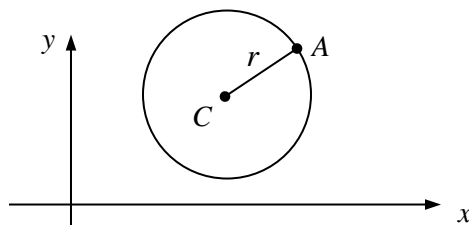
By the distance formula, $d(A, C) = \sqrt{(x - a)^2 + (y - b)^2} = r$. Square both sides, and get

Equation of the Circle:

$$(x-a)^2 + (y-b)^2 = r^2$$

This equation is called the equation of circle in **standard form**. Here (a, b) are coordinates of the center of the circle, and r is its radius.

Example 12.5. Identify the center and the radius of the circle $(x-3)^2 + (y+5)^2 = 15$.

Solution. This equation is given in standard form, and we immediately get the answer: the center has coordinates $(3, -5)$, and the radius is $\sqrt{15}$.

Note. Notice that in the above example, the second coordinate of the center is -5, not 5. This is because according to the equation of circle, we have $y-b = y+5 = y-(-5)$, so $b = -5$. Also, the radius is equal to $\sqrt{15}$, but not 15, since the number 15 is the square of the radius: $r^2 = 15$.

Equation of a circle may be given in a non-standard form. In this case, to identify the center and the radius of this circle, we represent the equation of circle in the standard form first. To achieve this, we can apply the completing the square technique. As a review, you may take a look at Session 10 "Completing the Square. The Quadratic Formula".

Example 12.6. Identify the center and radius of the circle given by the equation

$$x^2 + y^2 + 8x - 10y + 32 = 0.$$

Solution. We reorganize the terms and write the equation like this

$$(x^2 + 8x) + (y^2 - 10y) + 32 = 0.$$

Now complete the square for both x and y. According to the procedure for completing the square that we described in Session 10, in each pair of parentheses we divide coefficients for x and y by 2 and square them: $(8/2)^2 = 16$, $(-10/2)^2 = 25$. Then we add 16 and 25 to both sides of the equation:

$$(x^2 + 8x + 16) + (y^2 - 10y + 25) + 32 = 16 + 25 \implies (x+4)^2 + (y-5)^2 + 32 = 41,$$

$$(x+4)^2 + (y-5)^2 = 41 - 32 \implies (x+4)^2 + (y-5)^2 = 9.$$

We obtained the equation of the circle in the standard form. From here, the coordinates of the center are $(-4, 5)$ and the radius is 3.

Example 12.7. Let $A(-4,3)$ and $B(-6,9)$ be the endpoints of a diameter of a circle. Find the equation of the circle in the standard form. (Diameter of a circle is a line segment that passes through the center and whose endpoints lie on the circle).

Solution. First, let's find the coordinates of the center $C(a, b)$ of the circle. The center is the midpoint of the diameter AB. Using the Midpoint Formula, we have

$$a = \frac{-4+(-6)}{2} = \frac{-10}{2} = -5, \qquad b = \frac{3+9}{2} = \frac{12}{2} = 6.$$

So, the center has the coordinates $(-5, 6)$. Next, we will find the radius r of the circle. It is equal to the distance from the center of the circle to any point on the circle. To calculate the radius, we can take any of the given points: A or B. Let's take the point $A(-4,3)$ and calculate $r = d(C, A)$. Using the Distance Formula, we have

$$r = \sqrt{\left((-5)-(-4)\right)^2 + (6-3)^2} = \sqrt{(-5+4)^2 + 3^2} = \sqrt{(-1)^2 + 9} = \sqrt{1+9} = \sqrt{10} \implies r^2 = 10.$$

Now, using the coordinates $(-5, 6)$ of the center of the circle and the square of its radius, which is 10, we can write the equation of the circle in the standard form:

$$\left(x-(-5)\right)^2 + (y-6)^2 = 10, \text{ or } (x+5)^2 + (y-6)^2 = 10.$$

Example 12.8. Graph the circle from example 12.6 and label four points on the circle.

Solution. In example 12.6, we found that the center of the circle is $(-4, 5)$ and its radius is 3. We use this information to graph the circle via the following steps:

1) Plot the center $(-4, 5)$.

2) From the center, draw dotted horizontal and vertical lines.

3) Along these lines, count 3 units (which is radius) starting from the center in all four directions: up, down, left and right. Mark the four corresponding points as A, B, C, and D. These points lie on the circle.

4) Draw the circle through the points A, B, C, and D.
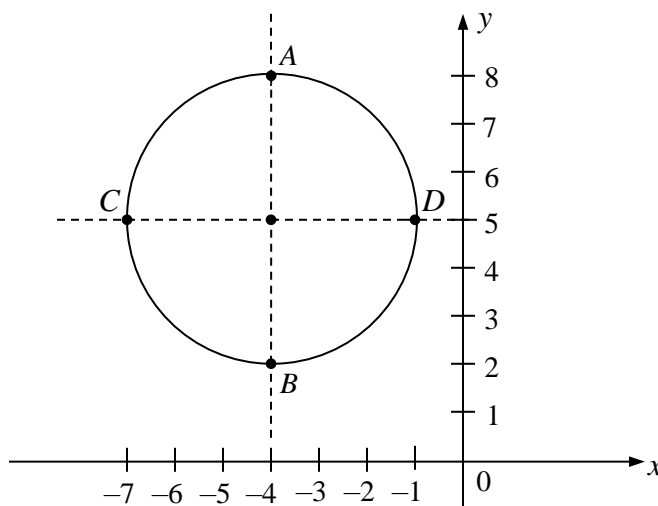
The points A, B, C, D have coordinates $A(-4, 8)$, $B(-4, 2)$, $C(-7, 5)$, and $D(-1, 5)$.

Exercises 12

In exercises 12.1 and 12.2, calculate the distance between the given points.

12.1. (5, 4) and (−1, 2)

12.2. (−2, 5) and (−3, 12)

In exercises 12.3 and 12.4, three vertices *A*, *B* and *C* of the triangle *ABC* are given.

1) Determine what kind of triangle it is: acute, obtuse, or a right triangle.

2) Determine which angle in the triangle *ABC* is the largest, and which angle is the smallest.

(You may want to review Example 12.2)

12.3. *A*(−2, −3), *B*(5, 2), *C*(6, −5)

12.4. *A*(−2, 4), *B*(−7, −1), *C*(4, −2)

In exercises 12.5 and 12.6, calculate the coordinates of the midpoint of the line segment with given endpoints.

12.5. (3, −4) and (−5, −6)

12.6. (4, −5) and (2, 7)

In exercises 12.7 and 12.8, *C* is the midpoint of the line segment *AB*. The coordinates of points *A* and *C* are given. Find the coordinates of the point *B*.

12.7. *A*(−9, 7), *C*(−4, 5)

12.8. *A*(5, 8), *C*(−1, 3)

In exercises 12.9 and 12.10, identify the coordinates of the center and the radius of the given circles.

12.9. a) $(x+2)^2 +(y+4)^2 = 36$

12.10. a) $(x-6)^2 +(y-3)^2 = 49$

b) $(x+5)^2 +(y-2)^2 = 20$

b) $(x-7)^2 +(y+8)^2 = 50$

In exercises 12.11 and 12.12, points *A* and *B* are given endpoints of a diameter of a circle. Find the equation of the circle in standard form $(x-a)^2 +(y-b)^2 = r^2$.

12.11. *A*(2, 8), *B*(−8, −16)

12.12. *A*(20, 25), *B*(4, −5)

In exercises 12.13 and 12.14, equations of circles are given. For each circle

1) Find the coordinates of the center and the radius.

2) Graph the circle.

3) Label four endpoints of vertical and horizontal diameters with their coordinates.

12.13. a) $x^2 + y^2 +4x -2y -11 = 0$

12.14. a) $x^2 + y^2 -8x +2y +8 = 0$

b) $x^2 + y^2 -10x -8y +32 = 0$

b) $x^2 + y^2 +6x +4y -12 = 0$

Session 13

Nonlinear Systems of Equations in Two Variables

We now consider examples of systems of two equations with two variables in which one or both equations are not linear. We also give a geometrical interpretation of that solutions.

Example 13.1. Solve the system of equations $\begin{cases} x + y = 6 \\ y = x^2 + 4x - 8 \end{cases}$

Solution. Notice that the first equation is linear, while the second is not (since it contains x^2). We can easily solve the first (linear) equation for x or y and substitute this expression into the second (nonlinear) equation. As a result, the second equation will contain only one variable. This method is called the **substitution** method. Just looking at the first equations, it seems that it does not matter for which variable to solve: for x or for y. However, it does matter for the second equation. If we solve the first equation for x, then we need to substitute this expression in the second equation for both x^2 and x, which is a bit involved. It is more suitable to solve the first equation for y. In this case we substitute this expression into the second equation for y, and we don't need to square any expression.

So, solve the first equation for y: $y = 6 - x$. Substitute this expression into the second equation, and get a quadratic equation in the variable x:

$$6 - x = x^2 + 4x - 8.$$

Solve this equation:

$$6 - x = x^2 + 4x - 8 \; \Rightarrow \; x^2 + 4x - 8 - 6 + x = 0 \; \Rightarrow \; x^2 + 5x - 14 = 0$$
$$\Rightarrow \; (x - 2)(x + 7) = 0 \; \Rightarrow \; x = 2 \text{ and } x = -7.$$

Now, using the expression $y = 6 - x$, find the corresponding values of y: if $x = 2$, then $y = 6 - 2 = 4$, and if $x = -7$, then $y = 6 - (-7) = 13$.
Final answer: the system has two solutions: $x = 2, y = 4$, and $x = -7, y = 13$. Or, as a solution set, $\{(2, 4), (-7, 13)\}$.

Note. We can interpret the above solutions geometrically using graphs of the given equations. The graph of the first equation is a straight line, and the graph of the second equations is a parabola. Solutions of the system are the coordinates of the intersection points of these two graphs. According to the final answer, the straight line and the parabola intersect each other at two points with the coordinates (2, 4) and (–7, 13). Here is the corresponding picture:
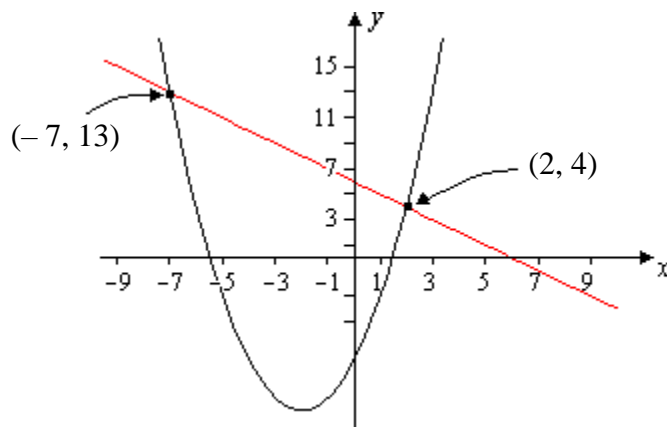
Example 13.2. Solve the system of equations $\begin{cases} y = x^2 - 3x + 14 \\ y = x^2 + 6x - 4 \end{cases}$.

Solution. In this example both equations are nonlinear, and both are solved for y. We can use different methods to solve this system. For example, we can equate the right sides of both equations. We can also eliminate y by subtracting one of the equations from the other. Using this method, subtract the first equation from the second and get

$0 = y - y = x^2 + 6x - 4 - (x^2 - 3x + 14) = x^2 + 6x - 4 - x^2 + 3x - 14$

$= 9x - 18$

$\Rightarrow \quad 0 = 9x - 18 \quad \Rightarrow \quad 9x = 18 \quad \Rightarrow \quad x = 2$.

To find y, we can substitute $x = 2$ into any equation of the system. Substituting it into the first equation, we get

$$y = 2^2 - 3 \cdot 2 + 14 = 4 - 6 + 14 = 12 .$$

Final answer: the system has one solution $x = 2$, $y = 12$, or as a pair (2, 12), or as a solution set $\{(2, 12)\}$.

Note. As in example 13.1, we can interpret the above solution geometrically. The graphs of both equations of the system are parabolas. According to the final answer, these parabolas intersect each other only at one point (2, 12):
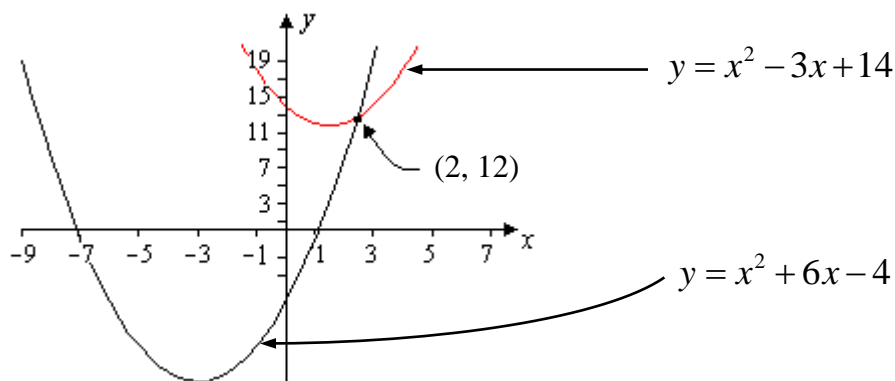
Example 13.3. Solve the system of equations $\begin{cases} 3x^2 + 2y^2 = 14 \\ 2x^2 + 5y^2 = 13 \end{cases}$.

Solution. Notice that in this system both variables, x and y, are in the second power only. We may temporarily use new variables u and v: $u = x^2$, $v = y^2$. Then, in terms of u and v, we have a linear system

$$\begin{cases} 3u + 2v = 14 \\ 2u + 5v = 13 \end{cases}$$

Solve this system by the elimination method (for example, eliminate the variable u):

$$\begin{array}{r} -2 \\ 3 \end{array} \begin{cases} 3u + 2v = 14 \\ 2u + 5v = 13 \end{cases} \Rightarrow \begin{cases} -6u - 4v = -28 \\ 6u + 15v = 39 \end{cases}.$$

Add the last two equations to eliminate u and solve for v: $11v = 11 \Rightarrow v = 1$. Substitute this value into the first equation of the system, and solve for u:

$$3u + 2 \cdot 1 = 14, \ \ 3u + 2 = 14, \ \ 3u = 12, \ \ u = 4.$$

So, $u = 4$ and $v = 1$. Now, we need to return from u and v to the original variables x and y. We have $x^2 = u = 4$. From here, $x = \pm\sqrt{4} = \pm 2$. So, we obtained two values for x: 2 and –2. Similarly, $y^2 = v = 1 \Rightarrow y = \pm\sqrt{1} = \pm 1$. We have two values of y: 1 and –1.

From this point, be very careful to write the final answer correctly. Any solution of the given system is a pair (x, y). Therefore, we need to combine each value of x with each value of y. As a result, the original system has four solutions:

$$\{(2, 1), \ (2, -1), \ (-2, 1), \ (-2, -1)\}.$$

Note. It can be shown that graphs of the equations in the given system are ellipses ("stretched" circles centered at the origin $(0, 0)$). The answer to the problem tells us that these two ellipses intersect each other at the above four points:
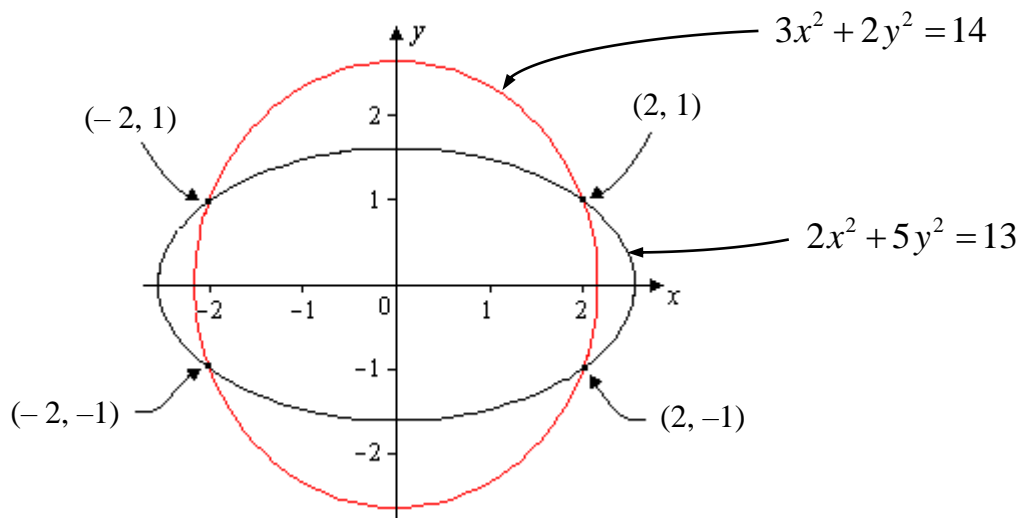
Example 13.4. Solve the system of equations $\begin{cases} y = \sqrt{x} \\ x^2 + 2y^2 = 15 \end{cases}$.

Solution. Here both equations are non-linear. Square the first equation: $y^2 = x$. Substitute this expression into the second equation and get a quadratic equation in x:

$$x^2 + 2x = 15 \implies x^2 + 2x - 15 = 0 \implies (x-3)(x+5) = 0.$$

From here we get two solutions of the quadratic equation: $x = 3$ and $x = -5$. Now we can use the first equation to find the corresponding values of y.

For $x = 3$, $y = \sqrt{3}$. So, one solution is the pair $\left(3, \sqrt{3}\right)$. For $x = -5$, $y = \sqrt{-5}$. In this session, we consider only real numbers. Because $\sqrt{-5}$ is not a real number, we reject it. Final answer: the original system has only one solution $\left(3, \sqrt{3}\right)$.

Note. Geometrically, the graph of the first equation is a curve, which has the shape of half of a parabola that "lies on its side": it is going not along the y-axis, but along the x-axis, and its graph is located above the x-axis. The second equation is the ellipse with the center in origin. Both curves intersect each other only at one point:
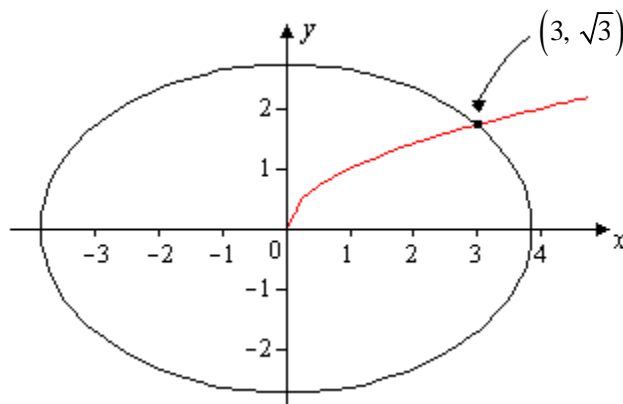
Example 13. 5. The area of a rectangular region is 96 square feet, and the perimeter is 40 feet. Find the dimensions of the region (i.e. find its length and width).

Solution. As for most word problems, we will solve it in two steps: set up the equations for the values in question, and then solve these equations. Also, check that the solution(s) make sense.

1) Let x be the length of the rectangle, and y be its width. Then $xy = 96$ (area of the rectangle), and $2x + 2y = 40$ (perimeter of the rectangle). We get the following system of equations:

$$\begin{cases} xy = 96 \\ 2x + 2y = 40 \end{cases}$$

2) We can simplify the second equation by dividing both sides by 2:

$$\begin{cases} xy = 96 \\ x + y = 20 \end{cases}.$$

Here the second equation is a linear one, and we can easily solve it for x or y. Let's solve it for y: $y = 20 - x$. Substituting this expression into the first equation, we obtain $x(20 - x) = 96$, or $20x - x^2 = 96$. This is a quadratic equation that can be rewritten in the standard form $x^2 - 20x + 96 = 0$. We can solve it by factoring:

$$(x - 8)(x - 12) = 0 \implies x = 8 \text{ and } x = 12.$$

We can get the corresponding value of y from the expression $y = 20 - x$. If $x = 8$, then $y = 20 - 8 = 12$. If $x = 12$, then $y = 20 - 12 = 8$.

Note. It looks like we have found two solutions: $x = 8$, $y = 12$, and $x = 12$, $y = 8$. These are two solutions of the system of equations. However, for the original problem that asks about the dimensions of the rectangle, these two solutions simply mean that one side of the rectangle is 8 feet, and the other is 12 feet. Assuming that the length is greater than the width, we come up to a unique solution.

Final answer: the length of the rectangle is 12 feet, and the width is 8 feet.

Exercises 13

In exercises 13.1 and 13.2, solve the systems of equations (find all real solutions).

13.1. a) $\begin{cases} 3x - y = 1 \\ x^2 + 2y = 5 \end{cases}$

13.2. a) $\begin{cases} 4x + y = 2 \\ x^2 + 3y = 19 \end{cases}$

b) $\begin{cases} x = y^2 + 5y - 2 \\ x = y^2 - 3y - 26 \end{cases}$

b) $\begin{cases} x = y^2 + 6y - 2 \\ x = y^2 + 3y + 16 \end{cases}$

c) $\begin{cases} \sqrt{x} + y = 0 \\ x^2 - 5y^2 = 36 \end{cases}$

c) $\begin{cases} \sqrt{x} - y = 0 \\ x^2 - 3y^2 = 4 \end{cases}$

d) $\begin{cases} x - 2y = 3 \\ x^2 - 3y^2 = -2 \end{cases}$

d) $\begin{cases} x + 3y = 2 \\ x^2 - 8y^2 = 4 \end{cases}$

e) $\begin{cases} 2x^2 + y^2 = 34 \\ 5x^2 + 3y^2 = 93 \end{cases}$

e) $\begin{cases} 5x^2 - 3y^2 = -28 \\ 3x^2 + 2y^2 = 44 \end{cases}$

f) $\begin{cases} 2x^2 - y^2 = -7 \\ 4x^2 + 3y^2 = 31 \end{cases}$

f) $\begin{cases} 3x^2 + y^2 = 37 \\ 5x^2 + 2y^2 = 70 \end{cases}$

In exercises 13.3 and 13.4, *A* represents the area of a rectangular region, and *P* represents its perimeter. Find the dimensions of the region (i.e. find its length and width, assuming that the width does not exceed the length).

13.3. $A = 12$ m², $P = 14$ m.

13.4. $A = 30$ yd², $P = 22$ yd .

Challenge Problems

13.5. Consider the system of equations $\begin{cases} x + y = a \\ x^2 + y^2 = b^2 \end{cases}$, where $a > 0$ and $b > 0$.

Prove the following statements:

1) If $a > b\sqrt{2}$, then the system does not have solutions.

2) If $a = b\sqrt{2}$, then the system has one solution.

3) If $a < b\sqrt{2}$, then the system has two solutions.

Interpret these results geometrically.

13. 6. Consider the system of equations $\begin{cases} \sqrt{x} - y = 0 \\ x^2 - (a+b)y^2 + ab = 0 \end{cases}$

Prove the following statements:

1) Let $a < 0$.
   a) If $b \geq 0$, then the system has one solution $\left(b, \sqrt{b}\right)$.
   b) If $b < 0$, then the system does not have solutions.

2) Let $a = 0$.
   a) If $b > 0$, then the system has two solutions $(0, 0)$ and $\left(b, \sqrt{b}\right)$.
   b) $b \leq 0$, then the system has one solution $(0, 0)$.

3) Let $a > 0$.
   a) If $b < 0$, then the system has one solution $\left(a, \sqrt{a}\right)$.
   b) If $b = 0$, then the system has two solutions $(0, 0)$ and $\left(a, \sqrt{a}\right)$.
   c) If $b > 0$ and $b = a$, then system has one solution $\left(a, \sqrt{a}\right)$.
   d) If $b > 0$ and $b \neq a$, then the system has two solutions $\left(a, \sqrt{a}\right)$ and $\left(b, \sqrt{b}\right)$.

13.7. Solve the system of equations $\begin{cases} ax - y = b \\ x^2 + cy = d(d+ac) - bc \end{cases}$.

Hint: Check that the quadratic equation $x^2 + c(ax-b) = d(d+ac) - bc$ has roots d and $-d-ac$.

Part II

Trigonometric Functions

Session 14

Geometric and Trigonometric (Oriented) Angles

Historically, trigonometry studies the relationships between angles and sides of triangles. In Greek, the word "Trigonometry" literally means "Triangle-Measurement".

It is important to understand that in geometry and trigonometry we treat angles in different ways.

In geometry, an angle is simply a figure that is formed by two rays coming from a common point, called the vertex of the angle.

Also, we assign the measure to an angle as some positive number. A common measure is the **degree measure**. If you cut a round pizza-pie (theoretically) into 360 equal slices, the angle in one slice is of one degree: $1°$.

Here are examples of angles (letter m means the measure of an angle):
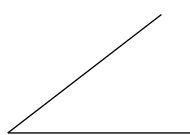
Acute Angle ($m < 90°$)   Right Angle ($m = 90°$)   Obtuse Angle ($m < 90°$)
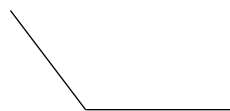
In trigonometry, we extend the meaning of an angle by assigning to it the "direction of rotation" and, as a result, the sign of its measure. That means that we assign to angles not only positive measure, but also negative. We can do this in the following way. Consider "geometric" angle

Terminal side
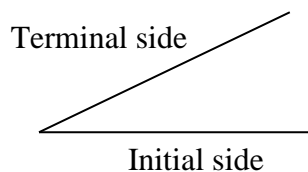
Initial side

Let's call one of its sides the **initial side**, and the other – **terminal side**. Let's say, the horizontal side is the initial, and the slant side is the terminal.

We can treat this angle as a result of rotation of the terminal side when it starts from the position of initial side and then rotates to its current position. To rotate, we have two directions: clockwise and counterclockwise. We can mark these two directions of rotation by arrows:
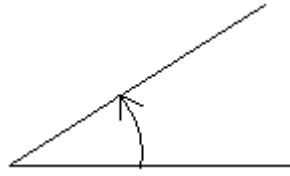
Counterclockwise rotation:
angle is positive.
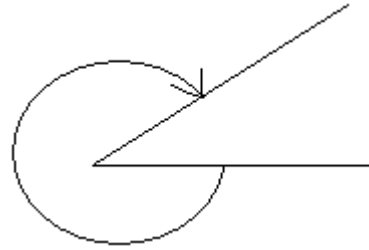
Clockwise rotation:
angle is negative.

As a matter of convention, a **positive** measure of an angle is assigned if the direction of rotation is counterclockwise, and a **negative** measure if the direction of rotation is clockwise. On the left figure above, the angle is positive, and on the right – negative. As you can see, taking just one "geometric" angle (two rays, coming from the same point), we can consider two "trigonometric" (or oriented) angles: one is positive, and another is negative depending on the direction in which we rotate the terminal side. Even more, we can assign to a given "geometric" angle infinitely many "trigonometric" angles making multiple full rotations of the terminal side in either direction. All such "trigonometric" angles have the same position of the terminal side, and they are called **coterminal angles**. In the figures above, the two angles are coterminal.

Example 14.1. Consider the angle of $40°$:
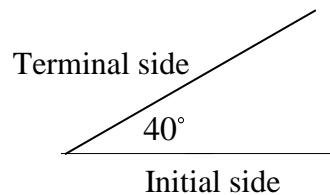
Describe all coterminal angles for this angle.

Solution. If we make one full rotation (rotation by $360°$) of the terminal side in either direction, the terminal side returns to its original position, and we obtain coterminal angle (i.e. the same "geometric" angle). We get the same result, if instead of one rotation, we make n full rotations (i.e. rotations by $360° \cdot n$). All such angles are coterminal to $40°$ angle and their values are described by the parametric formula $40° + 360° \cdot n$, where parameter n is any integer (positive, negative, or zero). We can write that $n = 0, \pm 1, \pm 2,...$. For positive n, we get positive values of the angle, and for negative n – negative values. For example, if we put $n = 1$ and $n = -1$, we get two specific coterminal angles: $40° + 360° = 400°$ and $40° - 360° = -320°$.

Two Special Right Triangles and Three Special Angles

In trigonometry, we often use the following two right triangles: one is a half of an equilateral triangle, and another is a half of a square:

$\triangle ABC$ is a half
of an equilateral triangle

$\triangle ABC$ is a half
of a square
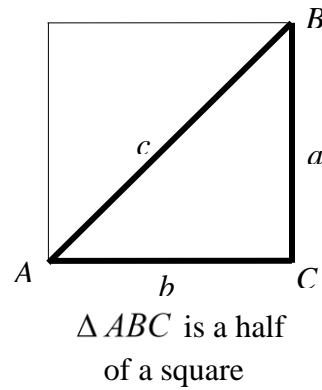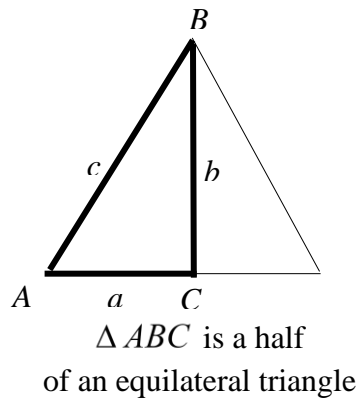
In the triangle *ABC* on the left picture, the acute angles are of $30°$ and $60°$. We will call such triangle a $30° - 60°$ triangle. In the triangle *ABC* on the right picture, both acute angles are of $45°$. We will call such triangle a $45° - 45°$ triangle. Both triangles: $30° - 60°$ and $45° - 45°$, are called **special right triangles**, and angles $30°$, $45°$ and $60°$ are called **special angles**.

Let's consider special triangles in more details. For both, we will use the **Pythagorean Theorem** that states that for any right triangle with the hypotenuse *c* and legs *a* and *b*, the following equation is true:

$$a^2 + b^2 = c^2.$$

We will also use this theorem in the forms:

$$c = \sqrt{a^2 + b^2}, \ a = \sqrt{c^2 - b^2}, \ b = \sqrt{c^2 - a^2}.$$

$30° - 60°$ Triangle

Let's draw the triangle like this
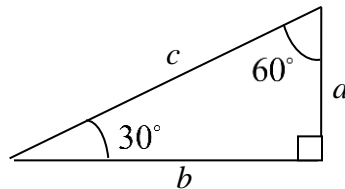
Recall that side *c* is the side of drawn above equilateral triangle, so side *a* is the half of side *c*: $a = \dfrac{c}{2}$ or $c = 2a$. Try to remember this fact:

In any $30° - 60°$ triangle, the leg opposite to $30°$ is the half of the hypotenuse (or the hypotenuse is twice as this leg).

Example 14.2. Consider $30° - 60°$ triangle with legs *a*, *b* and hypotenuse *c* (see picture above). Solve the following problems.

1) $a = 7$. Find b and c.
2) $b = 5$. Find a and c.
3) $c = 10$. Find a and b.

Solution. In all problems, side a is opposite to $30°$ angle. Therefore, $c = 2a$.

1) $c = 2a = 2 \times 7 = 14$. By the Pythagorean Theorem

$$b = \sqrt{c^2 - a^2} = \sqrt{14^2 - 7^2} = \sqrt{196 - 49} = \sqrt{147} = \sqrt{49 \cdot 3} = 7\sqrt{3}.$$

2) By the Pythagorean Theorem $c^2 = a^2 + b^2$ and $c = 2a$. Therefore,

$$c^2 = (2a)^2 = a^2 + b^2 \implies 4a^2 = a^2 + b^2 \implies 3a^2 = b^2 \implies a^2 = \frac{b^2}{3} = \frac{25}{3}.$$

$$a = \sqrt{\frac{25}{3}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}, \quad c = 2a = \frac{10\sqrt{3}}{3}.$$

3) Again, $c = 2a$, so $a = \dfrac{c}{2} = \dfrac{10}{2} = 5$. By the Pythagorean Theorem

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 5^2} = \sqrt{100 - 25} = \sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}.$$

Let's describe the connection between sides of $30° - 60°$ triangle in general form. Let a be a side, opposite to $30°$. Then the hypotenuse $c = 2a$. Another side b which is opposite to $60°$, can be calculated by the Pythagorean Theorem:

$$b = \sqrt{c^2 - a^2} = \sqrt{(2a)^2 - a^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}.$$

Solving this equation for a, we get

$$a = \frac{b}{\sqrt{3}} = \frac{b\sqrt{3}}{3}.$$

We can draw the following pictures
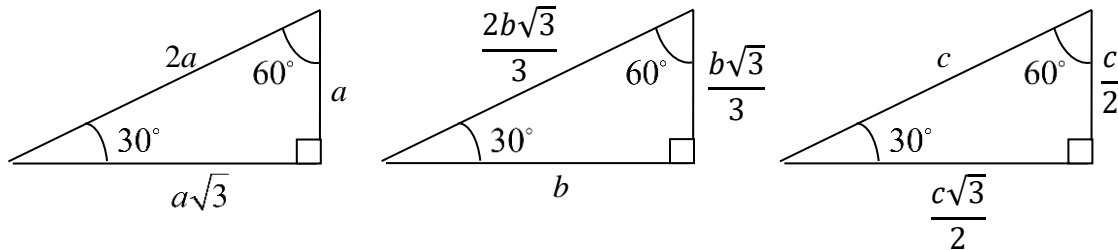
Using these pictures, you can solve problems like 14.2 faster.

$45° - 45°$ Triangle

The triangle looks like this
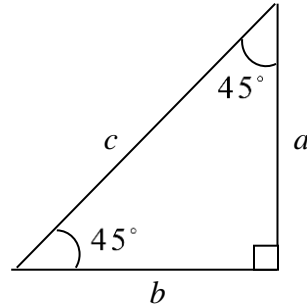
Both sides a and b are sides of the square above, therefore, they are equal: $a = b$. Try to remember this fact:

> In any $45° - 45°$ triangle both legs are equal.

Example 14.3. Consider $45° - 45°$ triangle with legs a, b and hypotenuse c (see picture above). Solve the following problems.

1) $a = 5$. Find b and c.

2) $b = 7$. Find a and c.

3) $c = 10$. Find a and b.

Solution. In all problems, a and b are two legs, so they are equal: $a = b$. Therefore, problems 1) and 2) actually the same (just numbers are different).

1) $a = b = 5$. By the Pythagorean Theorem

$$c = \sqrt{a^2 + b^2} = \sqrt{25 + 25} = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}.$$

2) $a = b = 7, \quad c = \sqrt{a^2 + b^2} = \sqrt{7^2 + 7^2} = \sqrt{7^2 \cdot 2} = 7\sqrt{2}.$

3) By the Pythagorean Theorem and using that $a = b$,

$$a^2 + b^2 = c^2, \ a^2 + a^2 = c^2, \ 2a^2 = 100, \ a^2 = 50, \ a = b = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}.$$

Similar to $30° - 60°$ triangle, let's describe the connection between sides of $45° - 45°$ triangle in general form. As we mentioned above, legs a and b are equal: $a = b$. The hypotenuse c can be calculated by the Pythagorean Theorem:

$$c = \sqrt{a^2 + b^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}.$$

Solving this equation for a, we get

$$a = \frac{c}{\sqrt{2}} = \frac{c\sqrt{2}}{2}.$$

We can draw the following pictures
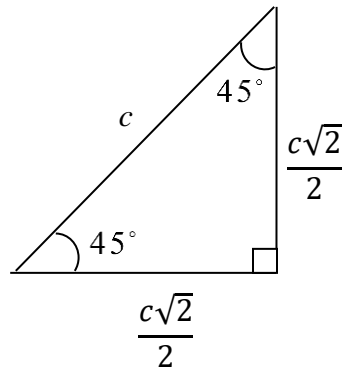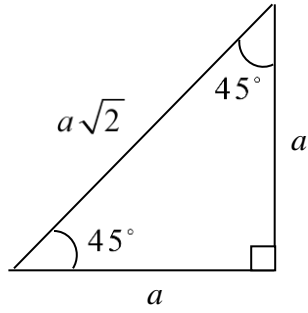
Exercises 14

In exercises 14.1 and 14.2, determine all the coterminal angles for angle θ. Also, indicate two positive and two negative particular coterminal angles (answers may vary).

14.1.
  a) $\theta = 50°$.

  b) $\theta = -70°$.

14.2.
  a) $\theta = 27°$.

  b) $\theta = -35°$.

In exercises 14.3 and 14.4, $30° - 60°$ triangle is given. In it, A, B and C are angles, and a, b, and c are sides, which are opposite to corresponding angles. $\angle A = 30°$, $\angle C = 90°$. Solve the given problems.

14.3.
  a) $a = 6$. Find b and c.

  b) $b = 3$. Find a and c.

  c) $c = 8$. Find a and b.

14.4.
  a) $a = 8$. Find b and c.

  b) $b = 9$. Find a and c.

  c) $c = 4$. Find a and b.

In exercises 14.5 and 14.6, $45° - 45°$ triangle is given. In it, A, B and C are angles, and a, b, and c are sides, which are opposite to corresponding angles. $\angle A = 45°$, $\angle C = 90°$. Solve the given problems.

14.5.
  a) $a = 4$. Find b and c.

  b) $b = 8$. Find a and c.

  c) $c = 9$. Find a and b.

14.6.
  a) $a = 3$. Find b and c.

  b) $b = 6$. Find a and c.

  c) $c = 7$. Find a and b.

Challenge Problem

14.7. A triangle (not necessary a right one) with **integer** sides, has a length of one side equal to 5, and the other – equal to 1. What is the length of the third side?

Session 15

Trigonometric Functions for Acute Angles

Definition of six trigonometric functions

Consider the following "giraffe" problem:

"A giraffe's shadow is 8 meters. How tall is the giraffe if the sun is $28°$ to the horizon?"

Trigonometric functions that we introduce here, allow us to solve this and many more problems that involve angles and sides of triangles. We will solve the above problem in the example 15.1 below.

To approach such problems, let's start with definition of trigonometric functions for acute angles.
Consider an acute angle θ:
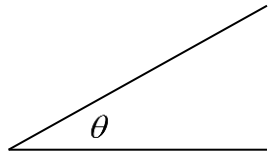
Trigonometric functions (in short trig functions) take this angle as its argument (as input) and assign some numerical values to it (output values). You will see shortly what exactly these values are.

Because angle θ is acute, we can always construct a right triangle with this angle:
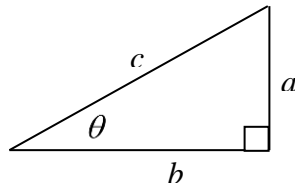
By proportionality properties of similar triangles, the ratios of the sides of this triangle do not depend on the size of the triangle; instead, they depend only on the value of the angle θ. In other words, if we take two right triangles with the same angle θ, but different sizes, then the ratios of the corresponding sides remain the same, since the triangles are similar. **Trigonometric functions** are exactly these **ratios**.

It is easy to see that there are only six possible ratios of the sides in a triangle. Here are all of them: *a/c*, *b/c*, *a/b*, *b/a*, *c/b*, *c/a*. So, there are exactly six trigonometric functions. Each of them has its own name and notation. The following table defines all six trig functions for angle θ.

Session 15: Trigonometric Functions for Acute Angles

Function Name	Function Notation	Definition
sine	$\sin \theta$	$\dfrac{a}{c}$
cosine	$\cos \theta$	$\dfrac{b}{c}$
tangent	$\tan \theta$	$\dfrac{a}{b}$
cotangent	$\cot \theta$	$\dfrac{b}{a}$
secant	$\sec \theta$	$\dfrac{c}{b}$
cosecant	$\csc \theta$	$\dfrac{c}{a}$

It may seem difficult to memorize all these functions. A simple advice (but perhaps not so easy to follow) is just to memorize them as you would the multiplication table.

From the above six trig functions, the first three are the most frequently used: sine, cosine, and tangent. They are called **basic** trig functions. The other three are reciprocals to basics: cotangent is the reciprocal to tangent, secant is the reciprocal to cosine, and cosecant is reciprocal to sine:

$$\cot \theta = \frac{1}{\tan \theta}, \ \sec \theta = \frac{1}{\cos \theta}, \ \csc \theta = \frac{1}{\sin \theta}.$$

A possible way to memorize that secant is the reciprocal to cosine, and cosecant is the reciprocal to sine is to remember the connection of the first letters in these words: letter c is related to letter s: $c \Leftrightarrow s$. Some people like the following mnemonic device **SohCahToa** for remembering the definition of the basic trig functions. It works like this. In the above right triangle, we can treat legs a and b as opposite and adjacent to the angle θ:
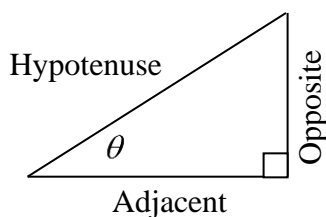
Now, the definition of sine, cosine, and tangent can be reformulated as

$$\begin{aligned} \sin \theta &= \text{Opposite/Hypotenuse} \\ \cos \theta &= \text{Adjacent/Hypotenuse} \\ \tan \theta &= \text{Opposite/Adjacent} \end{aligned}$$

The first three letters of the word SohCahToa mean: **S**ine is the ratio of **O**pposite leg to **H**ypotenuse, so we get **Soh**, and so on.

Trig Functions for Special Angles

In the previous session we have introduced three special angles $30°$, $45°$ and $60°$ as angles in special right triangles $30° - 60°$ and $45° - 45°$. Here we calculate the basic trig functions sine, cosine and tangent for these angles.

Because trig functions do not depend on the size of a triangle, for calculations, we can choose any value for one of the sides. Let's select the value of 1 for the shortest leg of $30° - 60°$ triangle and for both legs of $45° - 45°$ triangle. Recall that in $30° - 60°$ triangle, hypotenuse is twice as the shortest leg (this leg is opposite to $30°$ angle), so the hypotenuse is 2. Then by Pythagorean Theorem the other leg is $\sqrt{2^2 - 1^2} = \sqrt{3}$. For $45° - 45°$ triangle, hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$. We can draw the following two pictures
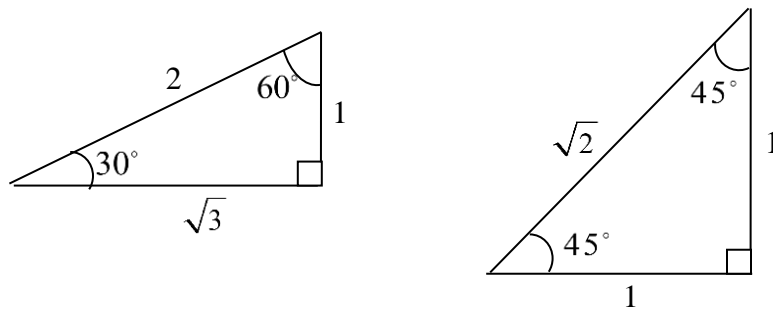
Now we use the definition of basic trig functions.

$30°$ angle: opposite side is 1, adjacent side is $\sqrt{3}$, and hypotenuse is 2. Therefore,

$$\sin 30° = \frac{1}{2}, \ \cos 30° = \frac{\sqrt{3}}{2}, \ \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

$60°$ angle: opposite side is $\sqrt{3}$, adjacent side is 1, and hypotenuse is 2. Therefore,

$$\sin 60° = \frac{\sqrt{3}}{2}, \ \cos 60° = \frac{1}{2}, \ \tan 60° = \sqrt{3}.$$

$45°$ angle: opposite side is 1, adjacent side is 1, and hypotenuse is $\sqrt{2}$. Therefore,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \ \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \ \tan 45° = 1.$$

We summarize these results in the following table

Angle θ	$30°$	$45°$	$60°$
$\sin \theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan \theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

To restore the values of $\sin \theta$ for the above special values and even for $\theta = 0$ and $\theta = 90°$ (we will define these values in the next session), the following expression can be used: $\frac{\sqrt{n}}{2}$. Just put $n = 0, 1, 2, 3$, and 4 according to the table

n	0	1	2	3	4
Angle θ	0°	30°	45°	60°	90°
$\sin \theta = \dfrac{\sqrt{n}}{2}$	$\dfrac{\sqrt{0}}{2} = 0$	$\dfrac{\sqrt{1}}{2} = \dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{4}}{2} = 1$

Working with arbitrary acute angles

The practical application of trigonometry is essentially based on the following general principal: it is easier to measure angles than distances. To measure angles, there is an optical device which is called the **clinometer**. Schematically, it looks like this:
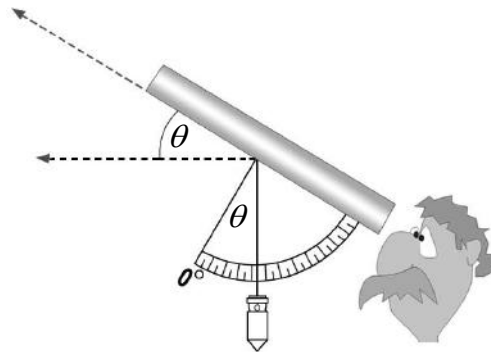
The angle from the horizontal line going up is called the angle of **elevation**. In the picture above, θ is the angle of elevation that can be measured by clinometer. In similar way, the angle from the horizontal line going down is called the angle of **depression:**

To find the values of basic trig functions for arbitrary angles, we can use buttons *sin*, *cos* and *tan* on scientific or graphing calculator.

Example 15.1. Let's solve the "giraffe" problem, stated at the beginning of this session. We can draw corresponding picture like this
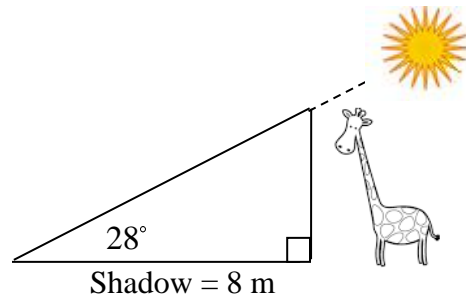
For the $28°$ angle, giraffe is the opposite side, and the giraffe's shadow is the adjacent side. A suitable trig function is tangent (ratio of the opposite side to adjacent). Let's denote giraffe's shadow by *s* and giraffe's height by *g*. We have $\tan 28° = \dfrac{g}{s}$. From here,

$$g = s \tan 28° = 8 \cdot 0.5317 = 4.25 \text{ m.}$$

Example 15.2. Nick launched a kite on a 120-m thread. The angle of elevation of the thread is $37°$. At what altitude is the kite flying?

Solution. Here is the corresponding picture
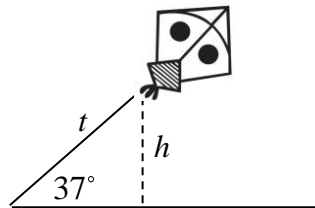
Let's denote the length of the thread by *t*. This is the hypotenuse and $t = 120$. The problem is to find height *h* which is opposite to the $37°$ angle. A suitable trig function is sine (ratio of the opposite side to hypotenuse). We have $\sin 37° = \dfrac{h}{t}$. From here,

$$h = t \sin 37° = 120 \cdot 0.6018 = 72.22 \text{ m.}$$

Example 15.3. A ladder is leaning against the wall such that the angle of depression of the top of the ladder is $56°$. What is the length of the ladder if the distance from its lower end to the wall is 2 m?
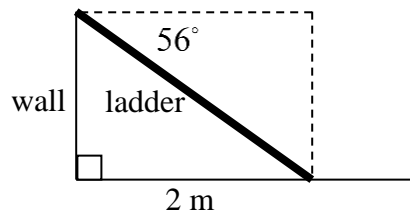
Solution. We draw two dotted line segments: one horizontal and one vertical. They have the same lengths as those to which they are parallel, in particular, horizontal line segment is 2 m. In the right triangle formed by the ladder and dotted lines, the ladder is the hypotenuse, and the top (horizontal) dotted line is the side adjacent to $56°$ angle. Let's

denote the length of this side by d and the ladder's length by l. We have $d = 2$ m. The problem is to find l. A suitable trig function is cosine (ratio of the adjacent side to hypotenuse). We have $\cos 56° = \dfrac{d}{l}$. From here,

$$l = \frac{d}{\cos 56°} = \frac{2}{0.559} = 3.58 \text{ m.}$$

Trig functions allow also to find angles in right triangles when info about sides is known. To solve such problems, first identify, similar to previous examples, which trig function relates to given problem and find the value of this function. Then, to find the angle you can use buttons $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ on calculator. These buttons calculate the values of so-called **inverse** trigonometric functions. These functions restore angles from the values of corresponding trig functions.

Example 15.4. A ship is 160 m away from the center of a horizontal barrier that measures 200 m from end to end. What is the minimum angle that the ship must be turned to avoid hitting the barrier?
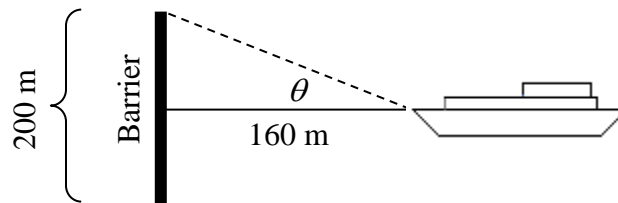
Solution. The problem is to find angle θ. Let's denote half of the length of barrier as a. We have $a = 200/2 = 100$ m. This is the side of right triangle on the picture and it is opposite to angle θ. Another side is the distance from the ship to the barrier. This side is adjacent to angle θ. We denote it as d. It is given that $d = 160$ m. Appropriate trig function here is tangent (ratio of the opposite side to adjacent).

$$\tan \theta = \frac{a}{d} = \frac{100}{160} = 0.625 \implies \theta = \tan^{-1}(0.625) = 32°.$$

Example 15.5. An airplane is flying at an altitude of 2.5 miles and is preparing for landing. It is 8.6 miles from the runway. Find the angle of depression that the airplane must make to land safely.
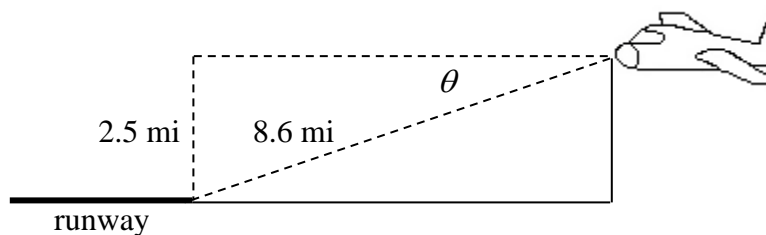
Solution. The problem is to find the angle θ. We denote the path (distance) of the airplane to the runway as d, and altitude as h. It is given that $d = 8.6$ mi and $h = 2.5$ mi. In the drawn right triangle (with dotted lines), d is hypotenuse, and h is opposite side for angle θ. Appropriate trig function here is sine (ratio of the opposite side to hypotenuse).

$$\sin\theta = \frac{h}{d} = \frac{2.5}{8.6} = 0.29 \implies \theta = \sin^{-1}(0.29) = 17°.$$

Example 15.6. Lillian wants to shingle her roof. The roofer asked her for the angle of elevation of the roof to make sure he can climb the roof safely. Help Lillian to calculate the angle according to the following picture:
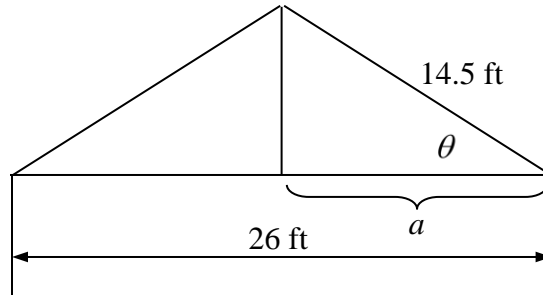
Solution. The problem is to find angle θ. Let's denote the marked horizontal line segment (half of the width of the house) as a, and slant line segment (the width of the roof) as s. We have $a = 26/2 = 13$ ft and $s = 14.5$ ft. In the triangle on the right side of the picture, s is hypotenuse and a is the side adjacent to angle θ. Appropriate trig function here is cosine (ratio of the adjacent side to the hypotenuse).

$$\cos\theta = \frac{a}{s} = \frac{13}{14.5} = 0.9 \implies \theta = \cos^{-1}(0.9) = 26°.$$

It looks like it is safe for the roofer to climb the roof.

Exercises 15

In problems 15.1 and 15.2, find sine, cosine and tangent of angles A and B.

15.1.
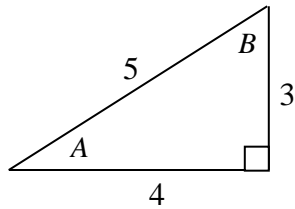
15.2
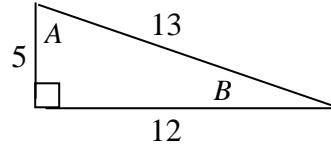
15.3. Given a right triangle with an acute angle θ, such that

$$a) \ \sin\theta = \frac{5}{13}, \qquad b) \ \cos\theta = \frac{6}{7}, \qquad c) \ \tan\theta = \frac{5}{4}.$$

Calculate other five trig functions.

15.4. Given a right triangle with an acute angle θ, such that

$$a) \ \sin\theta = \frac{3}{5}, \qquad b) \ \cos\theta = \frac{5}{6}, \qquad c) \ \tan\theta = \frac{3}{2}.$$

Calculate other five trig functions.

By solving a triangle, we mean to find its main elements: three sides and three angles, knowing some of them.

In problems 15.5 – 15.14, solve the given triangle.

15.5.

15.6.

15.7.

15.8.

15.9.

15.10.

15.11.

A ——8—— C
B
5

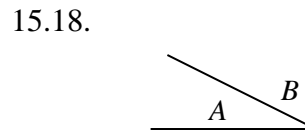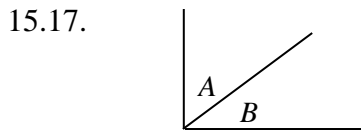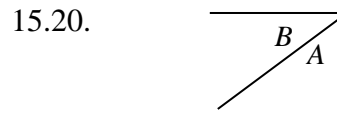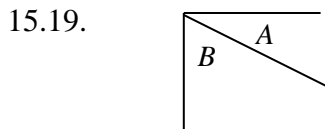15.12.

C ——9—— B
4
A

15.13.

C
7
A ——9—— B

15.14.

C ——11—— B
6
A
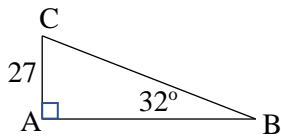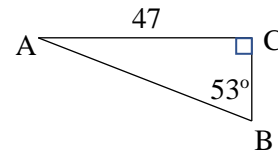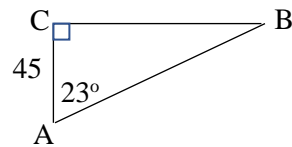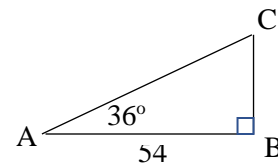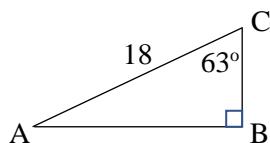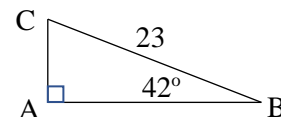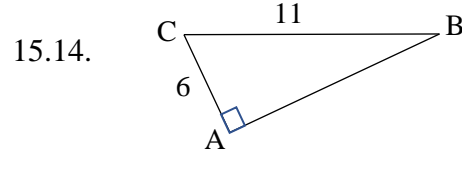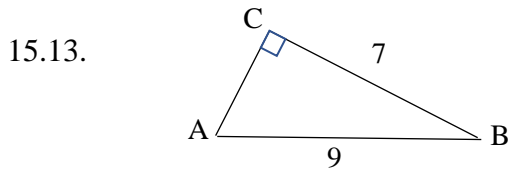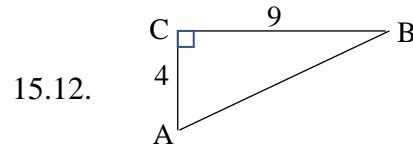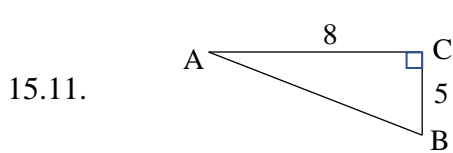
15.15. Find the exact values of secant and cosecant of angles $30°$, $45°$ and $60°$.

15.16. Find the exact value of cotangent of angles $30°$, $45°$ and $60°$.

In problems 15.17 and 15.18, determine which angle is the angle of elevation: *A* or *B*.

15.17.

A
B

15.18.

B
A

In problems 15.19 and 15.20, determine which angle is the angle of depression: *A* or *B*.

15.19.

A
B

15.20.

B
A

In the problems below, round answers to the nearest tenth.

15.21. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is $70°$. If the building is 80 ft tall, how far away is the fountain from the building?

15.22. Allison is looking at the top of a tall building. Her eyes are 5 feet above the ground. The angle of elevation is $75°$ and she is 15 feet from the building. How tall is the building?

15.23. A 20 foot ladder rests against a wall. Its angle of elevation from the ground is $55°$. How far from the wall is the base of the ladder?

15.24. Nick needs to reach a top window of the house using a ladder. He wants to put a ladder 1.5 meters from the wall. At this point, he measured that the angle of elevation to the window is $53°$. How long does the ladder have to be?

15.25. Ben is flying a kite and realizes that 260 feet of string are out. The angle of elevation of the kite is $40°$. How high is kite above the ground?

15.26. Lillian is swimming in the sea and notices a coral reef at the sea bottom. The angle of depression is $37°$ and the depth of the sea here is 50 feet. How far is she from the reef?

15.27. Over 6000 feet (horizontal), a road rises 330 feet (vertical). What is the angle of elevation?

15.28. Suppose a tree 15 m in height casts a shadow of length 27 m. What is the angle of elevation from the end of the shadow to the top of the tree?

15.29. A boat is sailing and spots a big shell 18 feet below the water. A diver jumps from the boat and swims 25 feet to reach the shell. What is the angle of depression from the boat to the shell?

15.30. A ladder leans against a wall. The foot of the ladder is 5.4 feet from the wall. The ladder is 15 feet long. Find the angle the ladder makes with the wall.

15.31. A vertical pole stands on the ground and has a support wire that runs from its top to the ground. The support is 50 feet long and anchored 22 feet from the base of the pole. Find the angle of elevation from the anchor point to the top of the pole.

15.32. Eli is putting up an antenna at the flat roof of a house. At its top, he attached a 50 ft guy wire and anchored it on the roof. Antenna is 30 ft long. What angle does the guy wire form with the antenna?

Challenge Problems

15.33. At a river shore (right near the water) a tree is standing. On the opposite side of the river, across the tree, Margaret is standing at the distance of 10 m from the water. She wants to determine the width of the river. She found that the angle of elevation to the top of the tree is $32°$. Then Margaret walked right to the water and found that the angle of elevation now became $43°$. What is the width of the river?

15.34. Esther and Nick stand at points E and N, 2 m apart in a dark room with a large mirror. Esther stands 2 m from the mirror, and Nick stands 1 m. At what angle EBA should Esther shine a flashlight on mirror so that the reflected light directly strikes Nick?

Note. According to the law of reflection, $\angle EBA = \angle NBC$.
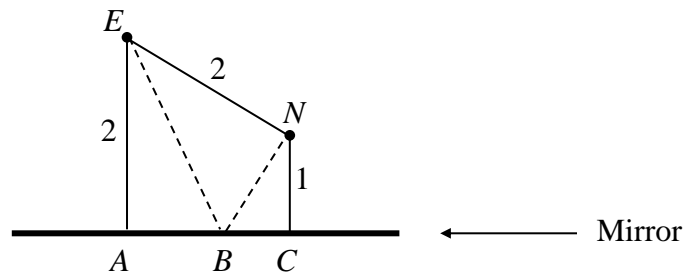
Session 16

# Trigonometric Functions for Arbitrary Angles

Unit Circle

In previous session we defined trig functions for acute angles: we constructed right triangle with given angle, and defined trig functions as the ratios of the sides in this triangle. This approach cannot be used for angles that are not acute like obtuse or negative angles: there are no right triangles with such angles.

Nevertheless, it is possible to define trig functions for arbitrary angles. To do this we will use a special tool that allows to reformulate definition of trig function of acute angles in such a way that a new definition can be used for arbitrary angles. This tool is called the **unit circle in the system of coordinates**.

This is just a circle with the radius of 1 and the center at the origin:
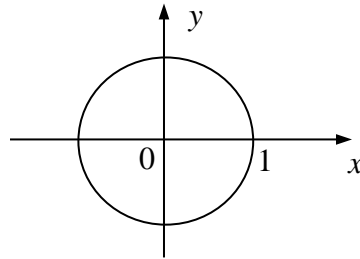

In this figure we will draw angles in the **standard position**. It means that their vertices are at the origin, and the initial sides goes along the positive part of the x-axis. Here is an example of such angle θ in the 1st quadrant (i.e. acute angle):
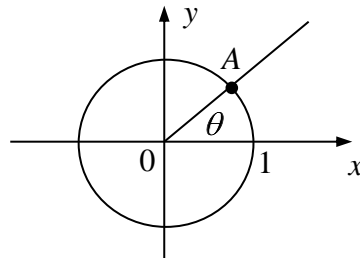
Angle θ is uniquely defined (up to coterminal angles) by the point A on the circle at which terminal side intersects the circle. We will call point A **corresponding** to the angle θ.

Let (a, b) be coordinates of the point A (we also use the notation $A(a, b)$ for point A):
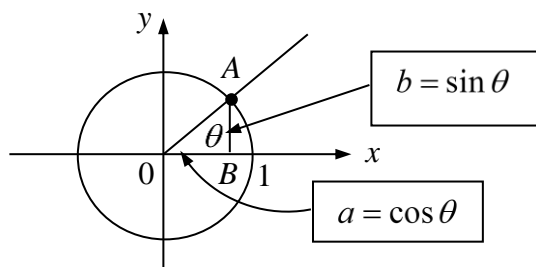
Notice that $0A = 1$ (radius of the unit circle). Then by the definition of thig functions in a right triangle $0AB$, we have

$$\sin\theta = \frac{AB}{0A} = \frac{b}{1} = b, \quad \cos\theta = \frac{0B}{0A} = \frac{a}{1} = a.$$

We see that for acute angles, **sine and cosine are the coordinates** of the corresponding points on the unit circle: sine is the second coordinate (y-coordinate), and cosine is the first coordinate (x-coordinate). We've got the reformulation (i.e. a new definition) of sine and cosine for acute angles: they are the coordinates of points on the unit circle. We can use this reformulation as a general definition for arbitrary angles.

Definition. Let θ be an arbitrary angle in the standard position, and $A(a, b)$ be the corresponding point on unit circle. Then, by definition,

$$\sin\theta = b, \quad \cos\theta = a, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

Note. To remember which of the trig functions – sine or cosine – is the first coordinate, and which is the second, you may use the alphabetical order of the first letters in the words **s**ine and **c**osine (c is before s, so **c**osine is the first coordinate, and **s**ine is the second).
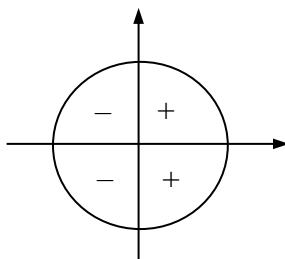
Other three trig functions can be defined as reciprocals to the basics:

$$\cot\theta = \frac{1}{\tan\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$
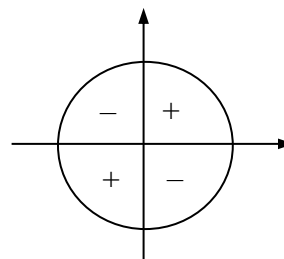
Because sine and cosine are coordinates, trig functions may take both positive and negative values depending on the quadrant in which angle θ lies. The following figures show the signs of basic trig functions.
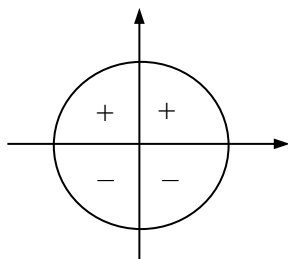

Signs of sine   Signs of cosine   Signs of tangent

Note. The following phrase may help you to remember which of these functions (sine cosine or tangent) is positive in each quadrant: "**A**ll **S**tudents **T**ake **C**alculus". This phrase hints that in the first quadrant all three are positive, in the second – only sine, in the third – only tangent, and in the fourth – only cosine.

Example 16.1. Calculate the basic trig functions for the quadrantal angles of $0°$, $90°$, $180°$, $270°$, and $360°$.

Solution.

Here is the picture for quadrantal angles:
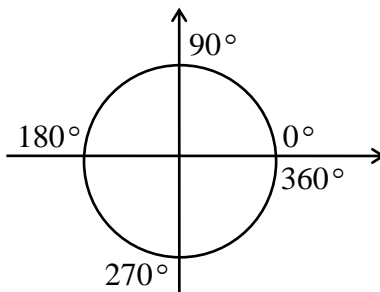
1) For $0°$ and $360°$ angles, the corresponding point on unit circle has coordinates $(1, 0)$. Therefore,
$\sin 0° = \sin 360° = 0$, $\cos 0° = \cos 360° = 1$, $\tan 0° = \tan 360° = 0$.

2) For $90°$ angle, the corresponding point has coordinates $(0, 1)$. Therefore, $\sin 90° = 1$, $\cos 90° = 0$. By definition, $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$. Because $\cos 90° = 0$, $\tan 90°$ is undefined (we cannot divide by zero).

3) For $180°$ angle, the corresponding point has coordinates $(-1, 0)$. Therefore, $\sin 180° = 0$, $\cos 180° = -1$, $\tan 180° = 0$.

4) For $270°$ angle, the corresponding point has coordinates $(0, -1)$. Therefore, $\sin 270° = -1$, $\cos 270° = 0$, $\tan 270°$ is undefined.

We summarize the results of example 16.1 in the following table

Angle θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin \theta$	0	1	0	−1	0
$\cos \theta$	1	0	−1	0	1
$\tan \theta$	0	undefined	0	undefined	0

We see that the maximal and minimal values of sine and cosine of the quadrantal angles are 1 and − 1 respectively. For all other angles sine and cosine are between –1 and 1. In general, for any angle θ

$$|\sin \theta| \leq 1, \ |\cos \theta| \leq 1.$$

There is no restriction for tangent.

Session 16: Trigonometric Functions for Arbitrary Angles

Reduction Formulas (The "Forehead Rule")

In example 16.1 we calculated sine, cosine and tangent for quadrantal angles $0°$, $90°$, $180°$, $270°$, and $360°$. Here we describe the way how to simplify sine and cosine of angles when we add (or subtract) angle θ to (from) quadrantal angles. In other words, we will simplify the following expressions:

$$\sin(90° \pm \theta), \ \cos(180° \pm \theta), \ \sin(270° \pm \theta), \ \sin(360° \pm \theta).$$

Formulas that simplify these expressions are called **reduction** formulas. For example, it is not difficult to get that $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$ (sine of an angle and cosine of complement angle are equal). Another example is $\cos(180° + \theta) = -\cos\theta$.

It is possible to analyze each of such expressions separately and get all reduction formulas (there are 16 of them in total). Instead, we suggest a simple rule to get (and memorize) such formulas. We call this rule the **Forehead Rule**.

Forehead Rule works like this. To get the reduction formula, assume that the angle θ is acute. We need to answer two questions:

1) Should we put a minus sign on the right side of the formula?

2) Should we change the sine to cosine and/or vice versa?

To answer the first question, determine the quadrant in which angle under consideration lies. Based on the quadrant, determine the sign of trig function (as described above).

To answer the second question, move your head along the axis on which the quadrantal angle lies. In doing this, you automatically get a "yes" or "no" answer.

Example 16.2. Get reduction formulas for

$$\sin(90° + \theta), \ \cos(180° + \theta), \ \sin(270° + \theta).$$

Solution.

For $\sin(90° + \theta)$:

1) Angle $90° + \theta$ lies in the 2nd quadrant. Here sine is positive, so minus sign is not needed.

2) Move your head along vertical axis (where $90°$ angle is located) and you get the answer "yes", so change sine to cosine. Final answer: $\sin(90° + \theta) = \cos\theta$.

For $\cos(180° + \theta)$:

1) Angle $180° + \theta$ lies in the 3rd quadrant. Here cosine is negative, so minus sign is needed.

2) Move your head along horizontal axis (where $180°$ angle is located) and you get the answer "no", so do not change cosine to sine. Final answer: $\cos(180° + \theta) = -\cos\theta$.

For $\sin(270° + \theta)$:

1) Angle $270° + \theta$ lies in the 4th quadrant. Here sine is negative, so minus sign is needed.

2) Move your head along vertical axis (where $270°$ angle is located) and you get the answer "yes", so change sine to cosine. Final answer: $\sin(270° + \theta) = -\cos\theta$.

Special cases of reduction formulas (when quadrantal angle is $0°$) are

$$\sin(-\theta) = -\sin\theta \text{ (odd property of sine)}$$
$$\cos(-\theta) = \cos\theta \text{ (even property of cosine)}$$
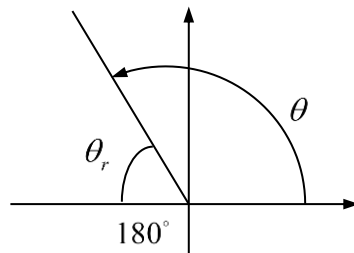
Reference Angle

This is a useful tool to reduce calculation of trig functions of arbitrary angles to acute angles.

Definition. Let θ be an arbitrary angle in standard position. Angle θ_r is called the **reference angle** to θ, if it satisfies three conditions:

1) Terminal side of θ_r coincides with the terminal side of θ.
2) Initial side of θ_r is horizontal (it coincides with either the positive or negative parts of the x-axis).
3) Angle θ_r is an acute positive angle.

Let's see how reference angle θ_r looks like depending on the quadrant in which original angle θ is located.

1) Angle θ is located in the first quadrant. Then θ_r coincides with θ: $\theta_r = \theta$.

2) Angle θ is located in the second quadrant. Then $\theta_r = 180° - \theta$:

3) Angle θ is located in the third quadrant. Then $\theta_r = \theta - 180°$:
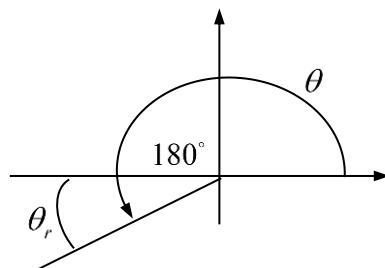
3) Angle θ is located in the fourth quadrant. Then $\theta_r = 360° - \theta$:

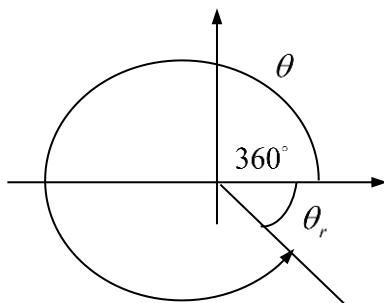A reference angle is useful because, up to a sign, the values of any trig function of θ coincide with the values of the same trig function for the reference angle θ_r, and θ_r is always an acute angle. You can check this using the reduction formulas described above.

Main Property of Reference Angle

> The absolute value of any trig function of any angle is equal to the value of the same trig function of the reference angle.

Hence, to calculate the value of a trig function, it is enough to find the sign of the function (based on the quadrant) and calculate the value of the trig function of the reference angle.

Example 16.3. Calculate $\cos 120°$.

Solution. Angle $120°$ is located in the 2$^{\text{nd}}$ quadrant, so $\cos 120° < 0$. This is the case 2) in the pictures above. Reference angle $\theta_r = 180° - 120° = 60°$. We have $\cos 60° = \dfrac{1}{2}$. Therefore,

$$\cos 120° = -\frac{1}{2}.$$

Example 16.4. Calculate $\sin 225°$.

Solution. Angle $225°$ is located in the 3$^{\text{rd}}$ quadrant, so $\sin 225° < 0$. This is the case 3) above. Reference angle $\theta_r = 225° - 180° = 45°$. We have $\sin 45° = \dfrac{\sqrt{2}}{2}$. Therefore,

$$\sin 225° = -\frac{\sqrt{2}}{2}.$$

Example 16.5. Calculate $\tan 330°$.

Solution. Angle $330°$ is located in the 4$^{\text{th}}$ quadrant, so $\tan 330° < 0$. This is the case 4) above. Reference angle $\theta_r = 360° - 330° = 30°$. We have $\tan 30° = \dfrac{\sqrt{3}}{3}$. Therefore,

$$\tan 330° = -\frac{\sqrt{3}}{3}.$$

Example 16.6. Find the values of the other five trig functions, if $\cos\theta = -\dfrac{5}{6}$ and $\tan\theta > 0$.

Solution. For reference angle θ_r, $\cos\theta_r = \dfrac{5}{6}$. Let's draw a right triangle, using the definition of $\cos\theta_r$ as the ratio of the adjacent leg to the hypotenuse:
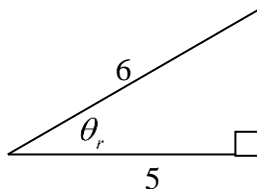
By the Pythagorean theorem, vertical leg of this triangle is $\sqrt{6^2 - 5^2} = \sqrt{11}$. From here, $\sin\theta_r = \dfrac{\sqrt{11}}{6}$ and $\tan\theta_r = \dfrac{\sqrt{11}}{5}$. Since $\cos\theta < 0$ and $\tan\theta > 0$, angle θ lies in the 3$^{\text{rd}}$ quadrant. Therefore, $\sin\theta = -\dfrac{\sqrt{11}}{6}$ and $\tan\theta = \dfrac{\sqrt{11}}{5}$. Other three trig functions are:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{6}{5}, \quad \csc\theta = \frac{1}{\sin\theta} = -\frac{6}{\sqrt{11}} = -\frac{6\sqrt{11}}{11}.$$

It is possible to define a trig function using a circle with an arbitrary radius r (not only a unit circle with $r = 1$). Namely, sine, cosine and tangent of any angle θ (in a standard position), which has point $A(a, b)$ on its terminal side are:

$$\boxed{\sin\theta = \frac{b}{r}, \quad \cos\theta = \frac{a}{r}, \quad \tan\theta = \frac{b}{a}, \quad r = \sqrt{a^2 + b^2}.}$$

Note. In the above formulas, the radius r is the distance from the point $A(a, b)$ to the origin.

Example 16.7. Find the value of the six trig functions of the angle θ if point $(2, -3)$ lies on the terminal side of angle θ, and θ is in the standard position.

Solution. We have $a = 2$, $b = -3$. Using the above formulas,

$$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (-3)^2} = \sqrt{13},$$

$$\sin \theta = \frac{b}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}, \quad \cos \theta = \frac{a}{r} = \frac{2\sqrt{13}}{13}, \quad \tan \theta = \frac{b}{a} = \frac{-3}{2} = -\frac{3}{2}.$$

Other three trig function are

$$\csc \theta = \frac{1}{\sin \theta} = -\frac{\sqrt{13}}{3}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{13}}{2}, \quad \cot \theta = \frac{1}{\tan \theta} = -\frac{2}{3}.$$

Exercises 16

In exercises 16.1 and 16.2, get the reduction formulas to the given expressions.

16.1.

a) $\sin\left(180° + \theta\right)$

b) $\cos\left(90° - \theta\right)$

16.2.

a) $\cos\left(180° - \theta\right)$

b) $\sin\left(270° - \theta\right)$

In exercises 16.3 and 16.4, find the reference angles to the given angles.

16.3.

a) $130°$

b) $320°$

c) $250°$

d) $85°$

e) $-130°$

16.4.

a) $200°$

b) $10°$

c) $310°$

d) $100°$

e) $-210°$

In exercises 16.5 and 16.6, reference angle of angle θ and the quadrant in which angle θ is located are given. Find the angle θ in the interval from $0°$ to $360°$.

16.5.

a) $40°$, quadrant III

b) $70°$, quadrant II

c) $50°$, quadrant IV

d) $20°$, quadrant I

16.6.

a) $40°$, quadrant IV

b) $70°$, quadrant I

c) $50°$, quadrant II

d) $20°$, quadrant III

In exercises 16.7 and 16.8,

1. Determine the quadrant in which angle is located.
2. Find the reference angle.
3. Calculate the exact value **without** using a calculator.

16.7.

a) $\sin 210°$

b) $\cos 300°$

c) $\tan 135°$

16.8.

a) $\sin 315°$

b) $\cos 150°$

c) $\tan 240°$

In exercises 16.9 and 16.10, use given information to determine the quadrant in which the angle θ is located and find the values of the five remaining trig functions.

16.9.

a) $\sin\theta = -\dfrac{2}{3}$ and $\tan\theta < 0$

b) $\cos\theta = -\dfrac{2}{5}$ and $\sin\theta > 0$

c) $\tan\theta = \dfrac{3}{5}$ and $\cos\theta < 0$

16.10.

a) $\sin\theta = \dfrac{4}{7}$ and $\cos\theta < 0$

b) $\cos\theta = -\dfrac{5}{8}$ and $\tan\theta > 0$

c) $\tan\theta = -\dfrac{7}{4}$ and $\sin\theta < 0$

In exercises 16.11 and 16.12, coordinates of a point are given. Find the values of six trig functions of an angle in the standard position for which the terminal side passes through this point.

16.11
  a) $(-1, -2)$
  b) $(4, -5)$
  c) $(-3, 7)$

16.12.
  a) $(-3, 5)$
  b) $(-5, 6)$
  c) $(-4, -7)$

Challenge Problems

16.13. Let θ be an acute angle. Which is greater: $\sin\theta$ or $\tan\theta$?

16.14. Prove that for any acute angle θ,

$$\sin\theta + \cos\theta > 1.$$

16.15. Working at a trig problem, Sophia calculated that $\sin\theta = \dfrac{1}{3}$ and $\cos\theta = \dfrac{2}{3}$. This result is wrong. Why?

16.16. Prove that if θ is not a quadrantal angle, then $\sin\theta$ and $\tan(\theta/2)$ have the same sign (both positive or both negative).

Session 17

Solving Oblique Triangles – Law of Sines

Oblique triangles are triangles that are not necessarily right triangles. We are going to "solve" them. It means to find its basic elements – sides and angles, given some of them. First of all, let's see what elements must be given. Obvious, if only angles are given and no sides, this info is not enough to determine sides since triangles with the same angles are similar and may have different sizes. So, at least one side must be given. We consider all possible cases when one, two or all three sides are given as well as some number of angles. More precisely, the following four cases are possible in solving triangles:

1) One side and two angles are given.

2) Two sides and an angle opposite to one of them are given.

3) Two sides and the angle between them are given.

4) Three sides are given.

Main tools to solve these problems are two important theorems: Law of Sines and Law of Cosines. In this session we consider Law of Sines and the first two problems.

Law of Sines

In any triangle, the bigger side, the bigger opposite angle. However, sides are **not** proportional to the opposite angles. For example, in the right triangle $30° - 60°$, if side opposite to $30°$ is a, then side opposite to $60°$ is $\sqrt{3}a$, which is not equal to $2a$. Law of Sines says that in any triangle sides are proportional to the **sines** of opposite angles. In other words, the ratio of any side to the sine of the opposite angle remains the same for all three sides in a given triangle.

More formally, the following theorem is true.

Theorem (Law of Sines). Consider triangle *ABC*:
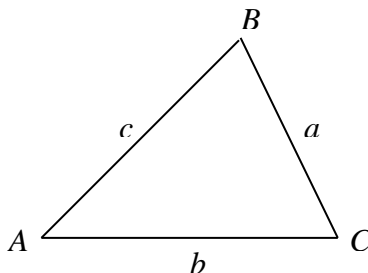
Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof. For simplicity, we consider only acute triangle (proof for obtuse triangle is slightly different, but similar). Let's draw the height h to the side b:

The height h splits the triangle ABC into two right triangles: ABD and BCD. Let's consider the sines of angles A and C:
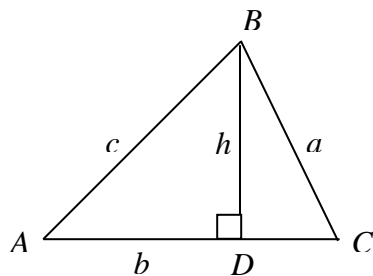
In triangle ABD, $\sin A = \dfrac{h}{c}$. Solve for h: $h = c \sin A$.

In triangle BCD, $\sin C = \dfrac{h}{a}$. Solve for h: $h = a \sin C$.

Equate the above two expressions for h: $c \sin A = a \sin C$. Divide both sides of this equation by $\sin A \cdot \sin C$ and get

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Similar ratio is true for the side b and angle B. The proof is completed.

Law of Sines works perfectly good for solving triangles for the case 1) above when a side and two angles of a triangle are given. In this case, the third angle is determined by subtracting the two given angles from $180°$, and then the Law of Sines is used to find the other two sides.

Example 17.1. Solve a triangle, if $a = 14$, $B = 40°$, and $C = 75°$.

Solution. We need to find angle A, and sides b and c.

1) $A = 180° - B - C = 180° - 40° - 75° = 65°$.

2) By the Law of Sines, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. From here, using calculator, we get

$$b = \frac{a \sin B}{\sin A} = \frac{14 \cdot \sin 40°}{\sin 65°} = 9.9.$$

3) Again, by the Law of Sines, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$. From here

$$c = \frac{a \sin C}{\sin A} = \frac{14 \cdot \sin 75°}{\sin 65°} = 14.9.$$

Final answer: $A = 65°$, $b = 9.9$, $c = 14.9$.

Using Law of Sines – Ambiguous Case

Now we consider the case 2) above: to solve a triangle when two sides and an angle opposite to one of them are given. In this case a triangle is not always defined uniquely, and we may face some difficulties to solve it. This is the ambiguous case. We will assume that the following data are given: sides a and b, and angle A opposite to side a (so, b is the adjacent side to angle A).

Case: angle A is obtuse

This is a simple case, since only two options are possible: the triangle does not exist, or the triangle is unique. To understand why, let's draw angle A and mark side b on its slanted side:
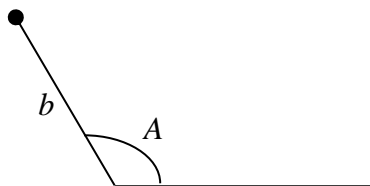
To get a triangle, we need to draw side a from the top point to meet with the horizontal side of angle A. Obviously, if side a is too short, it will not meet the horizontal side, and the triangle does not exist:
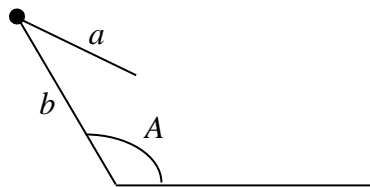
For a triangle to exist, side a must be greater than b. Then triangle is uniquely defined. We come up to the following result.

Proposition 17.1. Let two sides a and b, and **obtuse** angle A opposite to side a are given. Then

    1) If $a \leq b$, the triangle does not exist.

    2) If $a > b$, the triangle exists and unique.

Note. Conclusion in part 1) is also clear by the following reason: if $a \leq b$, then $A \leq B$. Since angle A is obtuse, then angle B must also be obtuse. But a triangle cannot have two obtuse angles.

Example 17.2. Solve a triangle, if $a = 18$, $b = 14$, and $A = 130°$.

Solution. We need to find angles B and C, and side c. Using the Law of Sines, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. From here

$$\sin B = \frac{b \sin A}{a} = \frac{14 \cdot \sin 130°}{18} = 0.596 \,.$$

Notice, that at this point we calculated **sine** of angle B, but not angle B itself. To restore the angle B from its sine, we can use the button $\sin^{-1}$ on a calculator similar to what we did in session 15 for right triangles. This button corresponds to the inverse sine. We have

$$B = \sin^{-1}(0.596) = 37° \,.$$

Now it is easy to find angle C: $C = 180° - A - B = 180° - 130° - 37° = 13°$.
To find side c, we can use Law of Sines again:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} = \frac{18 \sin 13°}{\sin 130°} = 5.3 \,.$$

Final answer: $B = 37°$, $C = 13°$, $c = 5.3$.

Case: angle A is acute

Similar to an obtuse angle, let's draw angle A and mark side b on its slanted side:
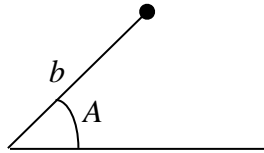
To create a triangle, we draw side a from the top point. There are four possible cases here:

1) Side a is too short to meet with the horizontal side:
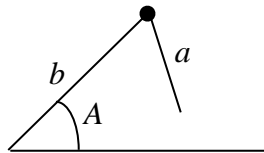
    Triangle does not exist.

2) Side a touches horizontal side exactly in one point:
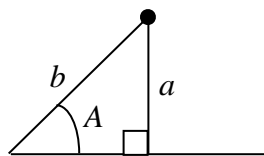
    We have a right triangle which is unique.

3) Side a intersects horizontal side in two points:
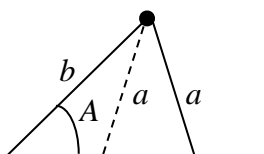
We have two triangles with sides a, b and angle A.

4) Side a is long enough, and intersects horizontal side only in one point to create a triangle:
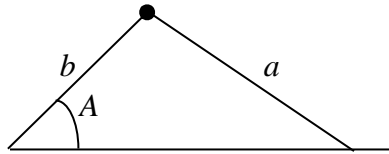
The triangle is unique. The top angle may be acute or obtuse.

How can we distinguish the above four cases using the values of sides a, b, and angle A? Take a look at this picture
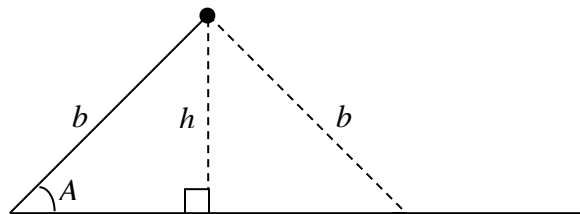
In your mind, draw side a from the top point. You can see that if $a < h$, side a is too short and the triangle does not exist. If $a = h$, we can draw only one right triangle. If a is between h and b: $h < a < b$, we can draw side a on both sides (left and right) of height h, and we have two triangles. Finally, if $a \geq b$, we can draw only one triangle. Notice that

$$\frac{h}{b} = \sin A \text{ , so } h = b \sin A \text{ .}$$

We come up to the following result.

Proposition 17.2. Let two sides a and b, and an **acute** angle A opposite to side a are given. Also, we denote $h = b \sin A$ (this is the height of the triangle, see the picture above). Four cases are possible:

1) If $a < h$, then the triangle does not exist.

2) If $a = h$, then the triangle is unique. It is a right triangle.

3) If $a \geq b$, then triangle is unique. This triangle may be acute, right, or obtuse (related to the top angle).

4) If $h < a < b$, then there are two triangles. One of them (whose side a is to the left of height h) is always obtuse, and the other (whose side a is to the right of height h) can be acute, obtuse, or right triangle (see exercise 17.28).

A practical way to use Proposition 17.2 is to directly apply the Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

and solve this equation for $\sin B$: $\sin B = \dfrac{b \sin A}{a}$. Three cases are possible when

calculating $\sin B$ by the above formula:

1) $\sin B > 1$. Because $\sin B$ cannot be greater than 1, triangle does not exist.

2) $\sin B = 1$. Then $B = \sin^{-1}(1) = 90°$ .The triangle is unique. It is a right triangle.

3) $\sin B < 1$. At least one triangle exists with the angle $B_1 = \sin^{-1}(\sin B)$. To see whether the second triangle exists, calculate the supplementary angle $B_2 = 180° - B_1$. If $B_2 + A < 180°$, then the second triangle exists with the angle B_2, otherwise, not.

Example 17.3. Let $b = 20$ and $A = 30°$. Determine the number of triangles that satisfy the given conditions. If the triangle exists, solve it.

    1) $a = 5$

    2) $a = 10$

    3) $a = 16$

    4) $a = 25$

Solution. Using the Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, we have $\sin B = \dfrac{b \sin A}{a}$. From a calculator (or just notice that $30°$ is a special angle), $\sin A = \sin 30° = 0.5$, and the expression for the $\sin B$ becomes $\sin B = \dfrac{20 \cdot 0.5}{a}$, so $\sin B = \dfrac{10}{a}$.

1) If $a = 5$, then $\sin B = \dfrac{10}{5} = 2$. Because sine cannot be greater than 1, triangle does not exist.

2) If $a = 10$, then $\sin B = \dfrac{10}{10} = 1$ and $B = \sin^{-1}(1) = 90°$. This is a right triangle. To solve it, calculate the angle C and side c.

    $C = 90° - A = 90° - 30° = 60°$. Side c can be found by the Pythagorean Theorem (notice that b is the hypotenuse, and a and c are legs):

    $c = \sqrt{b^2 - a^2} = \sqrt{20^2 - 10^2} = \sqrt{300} = 10\sqrt{3}$.

    Final answer: $B = 90°$, $C = 60°$, $c = 10\sqrt{3}$.

3) If $a = 16$, then $\sin B = \dfrac{10}{16} = 0.625$ and $B = \sin^{-1}(0.625) = 39°$. The other angle B', such that $\sin B' = \sin B$ is an obtuse angle supplementary to angle B: $B' = 180° - B = 180° - 39° = 141°$. We accept it because

$$B' + A = 141° + 30° = 171° < 180°.$$

Another reason to accept B' is that $b > a$ (see part 3 in "Case: angle A is acute"). So, we have two triangles. Let's solve them. It remains to find angle C and side c.

a) Triangle with angle $B = 39°$. We have

$$C = 180° - A - B = 180° - 30° - 39° = 111°.$$

By the Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \implies c = \frac{a \sin C}{\sin A} = \frac{16 \sin 111°}{\sin 30°} = 20.87.$$

b) Triangle with angle $B = 141°$ (we use letter B instead of B'). We have

$$C = 180° - A - B = 180° - 30° - 141° = 9°.$$

By the Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \implies c = \frac{a \sin C}{\sin A} = \frac{16 \sin 9°}{\sin 30°} = 5.01.$$

Final answer: There are two triangles:

$$B = 39°, \ C = 111°, \ c = 20.87.$$

$$B = 141°, \ C = 9°, \ c = 5.01.$$

4) If $a = 25$, then $\sin B = \dfrac{10}{25} = 0.4$ and $B = \sin^{-1}(0.4) = 24°$. Another angle B', such that $\sin B' = \sin B$ is supplemental to B and is obtuse angle:

$$B' = 180° - B = 180° - 24° = 156°.$$

We reject it because

$$B' + A = 156° + 30° = 186° > 180°.$$

Another reason to reject B' is that $b < a$, so angle B' cannot be obtuse (it should be less that angle A). So, we have only one triangle with angle $B = 24°$. To solve the triangle, it remains to find the angle C and the side c.

$$C = 180° - A - B = 180° - 30° - 24° = 126°.$$

By the Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \implies c = \frac{a \sin C}{\sin A} = \frac{25 \sin 126°}{\sin 30°} = 40.45 \quad .$$

Final answer: $B = 24°, \ C = 126°, \ c = 40.45$.

Exercises 17

Round the answers (where applicable) to the nearest tenth. For a triangle *ABC*, use the following notation: *A*, *B*, and *C* are angles, and *a*, *b*, and *c* are sides opposite to the corresponding angles. Similar notations are used for a triangle *PQR* with angles *P*, *Q* and *R*, and sides *p*, *q* and *r*, as well as for a triangle *KLM* with angles *K*, *L* and *M*, and sides *k*, *l* and *m*.

In problems 17.1 and 17. 2, solve a triangle *PQR* using the given information.

17.1. $r = 47$, $P = 50°$, and $Q = 110°$.　　　17.2. $p = 23$, $Q = 28°$, and $R = 67°$.

In problems 17.3 and 17. 4, solve a triangle *KLM* using the given information.

17.3. $k = 10$, $m = 15$, and $M = 40°$.　　　17.4. $l = 35$, $m = 20$, and $L = 70°$.

17.5. Three friends, Alice, Bob and Carol, are camping in their own tents on a flat meadow in the woodland. Alice and Carol are 25 m apart. The angle going from Alice to Bob and Carol is $20°$, and angle going from Bob to Alice and Carol is $110°$. How far apart are Bob and Carol?

17.6. Back to the previous problem. On the next trip, the three friends set up tents such that Alice and Bob are 15 m apart. The angle going from Bob to Alice and Carol is $70°$, and angle going from Carol to Alice and Bob is $40°$. How far apart are Alice and Carol?

17.7. On the next trip, the three friends set up tents such that Alice and Carol are 12 m apart. The angle going from Alice to Bob and Carol is $50°$, and angle going from Carol to Alice and Bob is $110°$. How far apart are Alice and Bob, and Bob and Carol?

17.8. A vertical post is standing on the ground and is supported by two wires (one on each side going in opposite direction). The ends of the wires on the ground are 7 ft apart. The angle of elevation of one of the wires is $80°$, and of the other wire is $65°$. Find the length of each wire.

17.9. Back to problem 17.5. On the next trip, the three friends set up tents such that Alice and Bob are 23 m apart, and Bob and Carol are 18 m apart. The angle going from Carol to Alice and Bob $55°$. What is the angle going from Alice to Bob and Carol?

17.10. On the next trip, the three friends set up tents such that Alice and Carol are 15 m apart, and Bob and Carol are 24 m apart. The angle going from Alice to Bob and Carol is $80°$. What is the angle going from Bob to Alice and Carol?

17.11. On the next trip, the three friends set up tents such that Alice and Bob are 17 m apart, and Bob and Carol are 25 m apart. The angle going from Alice to Bob and Carol is $50°$. What is the angle going from Bob to Alice and Carol?

17.12. On the next trip, the three friends set up tents such that Alice and Bob are 18 m apart, and Alice and Carol are 13 m apart. The angle going from Carol to Alice and Bob is $40°$. What is the angle going from Alice to Bob and Carol?

17.13. On the next trip, the three friends set up tents such that Alice and Carol are 12 m apart, and Alice and Bob are 19 m apart. The angle going from Carol to Alice and Bob is $110°$. How far apart are Bob and Carol?

17.14. On another trip, the three friends set up tents such that Alice and Carol are 19 m apart, and Bob and Carol are 25 m apart. The angle going from Alice to Bob and Carol is $50°$. How far apart are Alice and Bob?

17.15. Nina intends to purchase a parcel of land in the shape of a triangle (say, *ABC* with sides *a*, *b*, and *c*). She hired an assessor to measure the parcel. Nina told the assessor that it would be enough to provide her with minimum information such that she could calculate the remaining measurements (angles and sides) herself. The assessor submitted the following report: $a = 370$ ft, $A = 91°$, $b = 400$ ft. Recently Nina took a course of Trigonometry, and she decided to fire this assessor. Why?

17.16. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $B = 115°$. Nina decided to fire this assessor as well. Why?

17.17. Nina then hired a third assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $B = 65°$, $C = 40°$. Nina again decided to fire this assessor. Why?

17.18. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $b = 500$ ft, $C = 42°$. Nina again decided to fire this assessor. Why?

17.19. Nina hired yet another assessor. This assessor submitted the following report: $a = 370$ ft, $b = 500$ ft, $c = 120$ ft. Nina decided to fire this assessor too. Why?

Challenge Problems

17.20. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $b = 400$ ft. Nina decided to fire this assessor. Why?

17.21. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 71°$, $b = 350$ ft, $C = 54°$. Nina decided to fire this assessor. Why?

17.22. Nina hired another assessor. This assessor submitted the following report: $a = 370$ ft, $A = 61°$, $b = 400$ ft. This time Nina decided not to fire the assessor but

to request additional information to be able to calculate remaining angles and sides. What additional info (non-numeric) is needed?

17.23. Two radar stations are located 15 miles apart. They detect an aircraft between them. The angle of elevation measured by the first station is $20°$, and the angle of elevation measured by the second station is $40°$. Find the altitude of the aircraft.

17.24. Back to the previous problem, solve it in general form. Let d be the distance between radar stations, and let A and B be the angles of elevations to the aircraft from the stations. As before, the aircraft is between radar stations. Find the formula that expresses altitude of the aircraft in terms of A, B and d. In this case, it is possible to input this formula into a computer program to calculate the altitude instantly and track it during the aircraft flight between radar stations.

17.25. Back to problem 17.23. This time radar stations detect an aircraft on the right side of both stations. The distance between stations is 15 miles, and the angles of elevations to the aircraft are $20°$ and $70°$. Find the altitude of the aircraft.

17.26. Back to the previous problem, solve it in general form. Let d be the distance between the radar stations, and let A and B be the angles of elevations to the aircraft. As before, the aircraft is on the same side (left or right) of both radar stations. Find the formula that expresses altitude of the aircraft in terms of A, B and d.

17.27. A 2.5 m flagpole is NOT standing up straight on the ground. It is supported by two wires (one on each side going in opposite direction), each 3 m long. Both wires make a $55°$ angle with the ground. How far apart is each wire from the flagpole (on the ground)?

17.28. Let two sides a and b, and an acute angle A opposite to side a in a triangle ABC are given. Denote $h = b \sin A$. Assume that $h < a < b$. As we showed in Proposition 17.2, part 4, there are two triangles with the given sides a, b, and angle A. One of them, whose side a is to the left of height h, is always obtuse (see the picture located above the proposition 17.2). Prove the following statements about the second triangle whose side a is to the right of the height h:

a) Let angle $A < 45°$. Then

   1) If $a < b \tan A$, the triangle is acute.

   2) if $a = b \tan A$, the triangle is right.

   3) If $a > b \tan A$, the triangle is obtuse.

b) Let angle $A \geq 45°$, then the triangle is acute.

Session 18

Solving Oblique Triangles – Law of Cosines

In previous session, using the Law of Sines, we considered two problems on solving triangles from the total of four: when one side and two angles are given, and when two sides and an angle opposite to one of the sides are given.
Here we consider the remaining two problems:

1) Two sides and the angle between them are given.
2) Three sides are given.

For both problems, a triangle is unique and, therefore, we do not have an ambiguous case. Method to solve these problems is based on another important law in trigonometry: the Law of Cosines.

Note. Formally speaking, in problem 2) a triangle does not exist, if one of the sides is greater than or equal to the sum of the other two sides. We will assume that this case will not happen.

Law of Cosines

This law can be treated as generalization of the Pythagorean Theorem from right triangles to oblique ones.
Consider the triangle
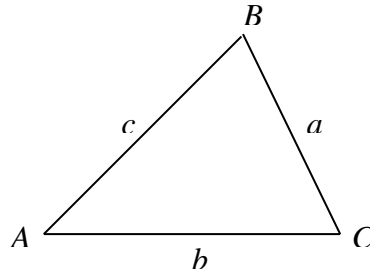
If C is not a right angle, we cannot conclude that $c^2 = a^2 + b^2$, so the Pythagorean Theorem is not true here. Instead, the following result is valid.

Theorem (Law of Cosines). For any triangle,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note. Consider the special case when $C = 90°$ (case of a right triangle). Then $\cos C = \cos 90° = 0$ and the above formula becomes $c^2 = a^2 + b^2$ which is exactly the Pythagorean Theorem. Therefore, the Law of Cosines can be considered as a generalization of the Pythagorean Theorem to oblique triangles.

Proof of the Law of Cosines.

Similar to the proof of the Law of Sines, we consider only case of acute triangles. Let's draw the height h to the side b:
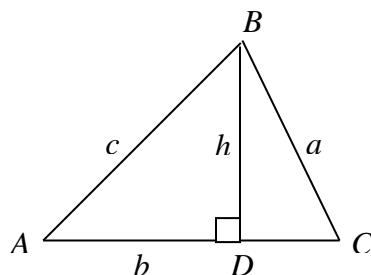
Height h breaks the triangle ABC into two right triangles: ABD and BCD. Let's write down the Pythagorean Theorem for each of them:

For the triangle ABD: $c^2 = AD^2 + h^2$. Notice that $AD = b - DC$. Therefore,

$$c^2 = (b - DC)^2 + h^2 = b^2 - 2bDC + DC^2 + h^2.$$

For the triangle BCD: $a^2 = DC^2 + h^2$.

Subtract the second equation from the first:

$$c^2 - a^2 = b^2 - 2bDC + DC^2 + h^2 - DC^2 - h^2 = b^2 - 2bDC.$$

Solving for c^2, we get

$$c^2 = a^2 + b^2 - 2bDC.$$

Now write down the definition of $\cos C$ from the triangle BCD: $\cos C = \dfrac{DC}{a}$. From here

$DC = a\cos C$. Substitute this expression into the above formula for c^2:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The theorem is proved.

Note. In this theorem, we have expressed side c through sides a, b and the angle C that is between them. Since all three sides play the same role, no one has any privileges over the others. Therefore, we can write similar expressions for the sides a and b:

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A \ \text{ and } \ b^2 = a^2 + c^2 - 2ac\cos B}$$

The Law of Cosines allows us to express cosine of any angle through three sides. To do this, just solve the above equations for cosines:

$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

As we mentioned, using the Law of Cosines we can solve triangles for the cases 1) and 2) indicated above. We will also use the property $A + B + C = 180°$.

Case 1. Two sides and the angle between them are given.

Example 18.1. Solve a triangle, if $a = 50$, $b = 15$, and $C = 55°$. Round the answers to the nearest tenth.

Solution. We need to find side c, and angles A and B.

1) By the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C = 50^2 + 15^2 - 2\cdot 50\cdot 15\cdot\cos 55°.$$

Using a calculator, $\cos 55° = 0.5736$ and

$$c^2 = 2500 + 225 - 1500\cdot 0.5736 = 1864.6.$$

$$c = \sqrt{1864.6} = 43.2.$$

2) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{15^2 + 1864.6 - 50^2}{2\cdot 15\cdot 43.2} = -0.3167.$

Using a calculator, $A = \cos^{-1}(-0.3167) = 108.5°$.

3) $B = 180° - A - C = 180° - 108.5° - 55° = 16.5°$.

Final answer: $c = 43.2$, $A = 108.5°$, $B = 16.5°$.

Note. In solving problems for Case 1, it is possible in step 2) to use the Law of Sines instead of the Law of Cosines. However, you need to be very careful when using button $\sin^{-1}$ on calculator. By pressing this button, you will always get **only an acute angle**, but the actual angle may be obtuse. To avoid possible mistake, we recommend that when using the Law of Sines to calculate an angle, **do not** start from the angle opposite the biggest side, because this angle may be obtuse. Always start from another angle that is definitely acute.

See, what may happen if you do not follow this advice. Let's return to Example 18.1, and try using the Law of Sines in step 2) to find the angle A, which is opposite to the largest side $a = 50$:

We have $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$. From here

$$\sin A = \frac{a\sin C}{c} = \frac{50\sin 55°}{43.2} = \frac{50\cdot 0.8192}{43.2} = 0.9481,$$

and, using a calculator, $\sin^{-1}(0.9481) = 71.5°$.

So, it looks like $A = 71.5°$. However, this answer is wrong. You can check it by calculating the angle $B = 180° - A - C = 180° - 71.5° - 55° = 53.5°$ and using the Law of Sines:

$$\frac{a}{\sin A} = \frac{50}{\sin 71.5°} = 52.7, \text{ but } \frac{b}{\sin B} = \frac{15}{\sin 53.5°} = 18.7.$$

The correct answer is the supplemental obtuse angle $108.5° = 180° - 71.5°$.

When using the Law of Cosines, you do not always start with c^2. **Start by solving for an unknown side for which the opposite angle is given**. The following example demonstrates it.

Example 18.2. Solve a triangle, if $b = 12$, $c = 15$, and $A = 25°$. Round the answers to the nearest tenth.

Solution. We need to find a, B and C.

1) Because angle A is given, we start by solving for its opposite side a:

$$a^2 = b^2 + c^2 - 2bc\cos A = 12^2 + 15^2 - 2\cdot 12 \cdot 15 \cdot \cos 25°.$$

Using the calculator, $\cos 25° = 0.9063$ and $a^2 = 144 + 225 - 360 \cdot 0.9063 = 42.73$.

$a = \sqrt{42.73} = 6.54$.

2) Let's find angle B using the Law of Sines. This is safe because its opposite side b is not the largest one, therefore, B is an acute angle (see Note above). We have

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \sin B = \frac{b\sin A}{a} = \frac{12\sin 25°}{6.54} = 0.775 \Rightarrow B = \sin^{-1}(0.775) = 50.8°.$$

3) $C = 180° - A - B = 180° - 25° - 50.8° = 104.2°$.

Final answer: $a = 6.5$, $B = 50.8°$, $C = 104.2°$.

Case 2. Three sides are given.

We only need to find three angles. Using the Law of Cosines, we can start with any angle. We recommend starting with the angle which is opposite to the biggest side. In doing this, we guarantee that the other two angles are acute, and to find them we can use either the Law of Cosines again or the Law of Sines (without making mistake mentioned in the Note above).

Here are our general recommendations:

> 1) When using the Law of Sines, start with the angle which
>   is opposite to the smallest side.
> 2) When using the Law of Cosines, start with the angle which
>   is opposite to the biggest side.

Example 18.3. Solve a triangle, if $a = 12$, $b = 20$, $c = 17$. Round the answers to the nearest tenth.

Solution. We need to find angles A, B and C.

1) According to the above recommendation, we use the Law of Cosines starting with the angle B which is opposite to the largest side $b = 20$.

$$b^2 = a^2 + c^2 - 2ac\cos B.$$

To find angle B, you can directly substitute given sides into this formula, or use the formula for $\cos B$, indicated right after the proof of the Law of Cosines. We will use this formula:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{12^2 + 17^2 - 20^2}{2 \cdot 12 \cdot 17} = 0.081, \ \ B = \cos^{-1}(0.081) = 85.4°.$$

2) To find angle A, let's use the Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \ \ \sin A = \frac{a \sin B}{b} = \frac{12 \sin 85.4°}{20} = 0.598, \ \ A = \sin^{-1}(0.598) = 36.7°.$$

3) $C = 180° - A - B = 180° - 36.7° - 85° = 57.9°.$

Final answer: $A = 36.7°$, $B = 85.4°$, $C = 57.9°$.

Example 18.4. Justify the following method to check whether a triangle with given sides a, b, and c is an acute, an obtuse or a right triangle:

Let c be the biggest side of the triangle. Calculate the value $E = a^2 + b^2 - c^2$.

    1) If $E > 0$, then the triangle is acute.

    2) If $E < 0$, then the triangle is obtuse.

    3) If $E = 0$, then the triangle is right.

Solution. Using the Law of Cosines, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{E}{2ab}.$

    1) If $E > 0$, then $\cos C > 0$ and $C < 90°$. Since c is a biggest side, C is a biggest angle. Therefore, two other angels are also less than $90°$ and the triangle is acute.

    2) If $E < 0$, then $\cos C < 0$ and $C > 90°$. The triangle is obtuse.

    3) If $E = 0$, then $\cos C = 0$ and $C = 90°$. The triangle is right.

A specific example of using this method was given in session 12, example 12.2.

    Note. If three sides of a triangle are given, you can use the result of Example 18.4 to check what kind of triangle you have before calculating the angles.

Exercises 18

Round answers (where applicable) to the nearest tenth. For a triangle PQR, use the following notation: P, Q, and R are angles, and p, q, and r are sides opposite to the corresponding angles. Similar notations are used for a triangle KLM with angles K, L and M, and sides k, l, and m.

In problems 18.1 and 18. 2, solve the triangle PQR using the given information.

18.1. $q = 20$, $r = 30$, and $P = 65°$. 18.2. $p = 35$, $r = 70$, and $Q = 130°$.

In problems 18.3 and 18. 4, solve the triangle KLM using the given information.

18.3. $k = 10$, $l = 15$, $m = 20$. 18.4. $k = 12$, $l = 8$, $m = 5$.

18.5. Eli and Ben came to the forest to pick some wild flowers. They started from the same point, and each walked in a straight line at an angle of $40°$ relative to each other. Every minute they call out to each other to avoid being lost. A sound in this forest can be heard from up to 60 m away. After 10 minutes, Eli has walked 80 m and Ben has walked 70 m. Can they hear each other at that time?

18.6. Lillian wants to measure the distance between two trees that are on opposite sides of a small pond. She started at one of the trees and walked 240 ft in a straight line along the pond. Then she turned at $115°$ toward the second tree and walked another 310 ft until she reached the second tree. What is the distance between the trees?

18.7. Three friends, Alice, Bob and Carol, are camping in their own tents on a flat meadow in the woodland. Alice and Carol are 10 m apart, and Alice and Bob are 15 m apart. The angle going from Alice to Bob and Carol is $50°$. What is the angle going from Bob to Alice and Carol?

18.8. Back to previous problem. On the next trip, the three friends set up tents such that Alice and Bob are 15 m apart, Bob and Carol are 20 m apart, and the angle going from Bob to Alice and Carol is $110°$. Find the angle going from Carol to Alice and Bob.

18.9. A small airplane is 60 miles from the airport and is going down for landing with some angle of depression. However, its navigation device malfunctions and incorrectly shows the distance to be 65 miles. The dispatcher noticed this mistake and figured out that if airplane continues on its current course, it will end up 16 miles from airport. By how many degrees should dispatcher adjust the airplane heading?

18.10. An aircraft is making a flight to airport *A*. At some point a pilot receives information that due to bad weather, airport *A* is closed, and he needs to fly to airport *B*. At that moment, the aircraft is 520 mi apart from airport *A*, and is 650 mi apart from airport *B*. The distance between airports *A* and *B* is 570 mi. By what angle should the pilot change the course of the aircraft to fly to airport *B*?

Challenge Problems

18.11. After hurricane Sandy, a small tree was leaning. To keep it from falling, it was nailed by a 7-foot strap into the ground 5 feet from the base of the tree. The strap was attached to the tree 4 feet along the tree from the ground. By what angle from the vertical position was the tree leaning?

18.12. The architect wants to design a bridge between points A and B on the two banks of a triangular pond (see the diagram below). He needs to know the length of the bridge. To calculate, he made measurements as shown below. What is the length of the bridge *AB*?
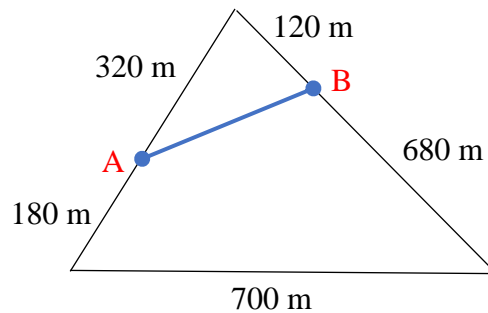
Session 19

Radian Measure of Angles

Most people familiar with the degree measure of angles. We already mentioned in session 14 that if we cut a round pizza pie (theoretically, of course) into 360 slices, the angle in one slice is of one degree (and this is a very tiny piece, so almost nothing to eat). But why the number 360 is used for the degree measure of angles?

This number was introduced by astronomers in ancient Babylon (at least 3000 B.C.). No one knows for sure why they settled for this number. At those times, it was already known that the yearly cycle consists of 365 and 1/4 days, even though astronomers didn't know yet that the earth revolves around the sun. It is reasonable to guess that they just rounded 365 and 1/4 to 360 because the number 360 has many more divisors. In other words, the number 360 can be divided into whole parts much better than 365. From this point of view, we could treat one degree angle as one day related to entire year. In any case, it's clear that angle measure based on the number 360 is artificial. It looks similar to the decimal system which is also an artificial one since it was introduced only because we have 10 fingers on our hands. In math, and especially in computer science, it is used more convenient systems like binary or octal which have as bases powers of two. These systems could be considered as natural ones.

And how about measurement of angles? Does some kind of natural measure of angles exist? The answer is "yes". This measure is called the **radian measure**.

To define the radian measure, consider an angle as a central angle in a circle. It means that we draw a circle and put the vertex of the angle in its center:
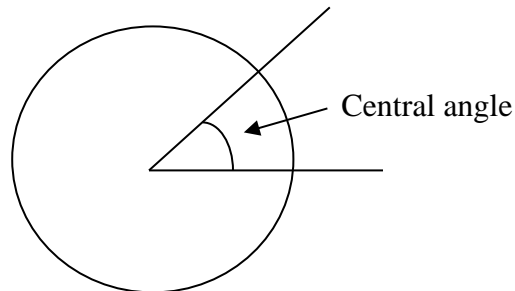
Of course, we can draw infinite many such circles. One of them is a **unit circle** (its radius is equal to 1). Using it, the **radian measure** (denoted as θ) of the central angle in unit circle is the **length s of the corresponding arc** (arc between two radii):

$$\theta = s$$
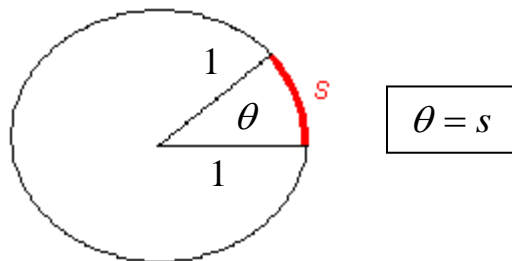
For any other circle (with arbitrary radius), by the proportionality, the ratio of the arc to the radius equals to the above arc of the unit circle. We come up to the following definition for arbitrary circle.

Definition of Radians. Consider an angle as a central angle: we draw a circle with the center in its vertex. Let the radius and the corresponding arc of the circle be r and s accordingly. Then the radian measure θ of the angle is defined as the ratio of s to r:

$$\theta = \frac{s}{r}$$
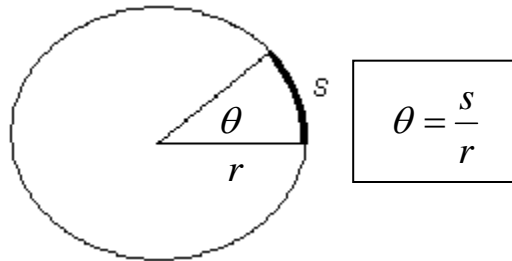
From here, $s = \theta \cdot r$. We may say that the radian measure of a central angle is the number of radii that can fit in the corresponding arc; hence the term "radian".

In particular, a central angle is of **one radian measure**, if the length of the corresponding arc is equal to the radius: $s = r$.
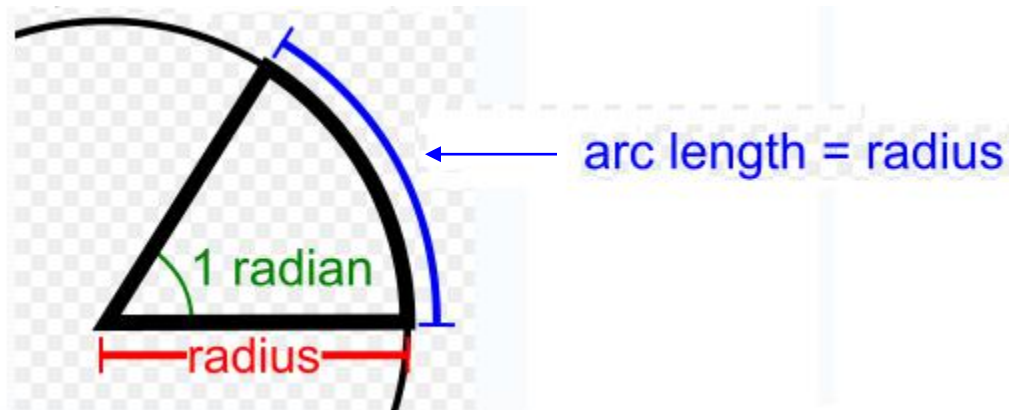
We may also say that a one-radian angle is an angle in a "curvilinear" equilateral triangle (sector) in which two sides are radii, the third side is an arc, and all three sides are equal.

From this point of view, it is easy to estimate the value of one radian. As we know, in a "normal" equilateral triangle all three angles are of $60°$. In "curvilinear" equilateral triangle, the central angle should be a bit less than $60°$ because the opposite side is an arc (a curve). Below in example 19.1 we will calculate that 1 radian $\approx 57.3°$. As we see, it is much better to cut our pizza pie by radians. In this case at least 6 people ($360/57.3 \approx 6$) will have something to eat.

At the first glance the radian measure may look a bit more complicated than the degree measure. However it is more useful in many problems in mathematics and science.

To understand the benefit of radian measure, let's re-write the above formula $\theta = \dfrac{s}{r}$ as $s = \theta \cdot r$. As you see, using the radian measure, the connection between arc, angle and radius is very simple. For any other measure of angles (for example, for degrees) this connection is more complicated and has the form $s = k \cdot \theta \cdot r$, where k is some numerical coefficient (we will show in example 19.4 below that for the degree measure, $k \approx 0.017$). Radian measure is different from all others by the simplest value $k = 1$. The main idea of the radian measure is to relate linear (length of the arc) and angular measurements in the simplest possible way. That's why many mathematical and technical calculations are simpler when using radians.

The idea of measuring angles by the length of the arc is credited to Roger Cotes in the early 1700s, an English mathematician who worked closely with Isaac Newton. But the term radian was first introduced only in the late 1800s by James Thomson, Ireland.

Let's set up a connection between the radians and degrees. Consider the angle of $360°$. This angle corresponds to a full rotation around a circle. If we consider it as a central angle, the corresponding arc s is the entire circumference. Recall the formula for the circumference of a circle: $s = 2\pi \cdot r$. Compare this formula with the above $s = \theta \cdot r$. By equating both, we get $\theta \cdot r = 2\pi \cdot r$. From here, $\theta = 2\pi$. We see that the angle $360°$ corresponds to 2π radians. This connection allows to express any degree measure in radians and vice versa. In particular, $180°$ corresponds to π radians. For any angle, let's denote its degree measure as $\theta°$, and the radian measure as θ_r. It is easy to set up a connection between $\theta°$ and θ_r using the proportion: $180°$ relates to π as $\theta°$ relates to θ_r

$$\boxed{\dfrac{180°}{\pi} = \dfrac{\theta°}{\theta_r}}$$

Let's call this proportion the **main proportion**.

By cross-multiplication, we get $180° \cdot \theta_r = \pi \cdot \theta°$. From here we can express $\theta°$ through θ_r and vice versa:

$$\theta° = \dfrac{180°}{\pi} \cdot \theta_r, \qquad \theta_r = \dfrac{\pi}{180} \cdot \theta°.$$

Note. You do not need to memorize these formulas. Just remember that $180°$ corresponds to π radians:

$$\boxed{180° = \pi_{rad}}$$

and then use the main proportion.

Example 19.1. Express the angle of 1 radian in degrees.

Solution. The main proportion takes the form

$$\frac{180°}{\pi} = \frac{\theta°}{1_r}$$

By cross-multiplication, $\theta° \cdot \pi = 180°$. From here, $\theta° = \frac{180°}{\pi} \approx \frac{180°}{3.14} \approx 57.3°$.

So, 1 radian $\approx 57.3°$.

Note. If angle in radians is given in terms of π, there is no need to use proportion to convert this angle into degrees: simply replace π with 180. In this way we can say immediately that $\frac{\pi}{2}$ is $90°$, $\frac{3\pi}{2}$ is $270°$, 2π is $360°$ and so on.

Example 19.2. Express the angle of $\frac{5\pi}{12}$ radians in degrees.

Solution. Replace π with 180 and you are done: $\frac{5\pi}{12} = \frac{5 \cdot 180°}{12} = 75°.$

Example 19.3. Express the angle of $1°$ in radians.

Solution. The main proportion takes the form

$$\frac{180°}{\pi} = \frac{1°}{\theta_r}$$

By cross-multiplication, $180 \cdot \theta_r = \pi$. From here, $\theta_r = \frac{\pi}{180} \approx \frac{3.14}{180} \approx 0.017.$

So, $1° \approx 0.017$ radians.

Example 19.4. Express the arc length of a central angle through the radius of the circle and the degree measure of the angle.

Solution. Let s, r, and $\theta°$ be the arc length, radius, and degree measure of the central angle accordingly. Also, denote by θ_r the radian measure of the angle. As we mentioned above, $s = \theta_r \cdot r$, and $\theta_r = \frac{\pi}{180} \cdot \theta°$. From here, $s = \frac{\pi}{180} \cdot \theta° \cdot r$. Using calculation $\frac{\pi}{180} \approx 0.017$, we can write the approximate formula $s \approx 0.017 \cdot \theta° \cdot r$.

Note. Let's recall once again that for radian measure, the connection between the arc length s, radius r, and the central angle θ_r is the simplest:

$$s = \theta_r \cdot r$$

For any other measure this relation is more complicated, for example for degrees, $s \approx 0.017 \cdot \theta° \cdot r$.

Using the main proportion, we can calculate the radian measure of special angles $30°$, $45°$ and $60°$, as well as of the quadrantal angles $0°$, $90°$ $180°$ $270°$ $360°$. The following table summarizes the calculations.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π

In conclusion, let's mark quadrantal angles in degrees and radians on the unit circle. Compare left and right figures.

Angles in degrees   **Angles in radians**
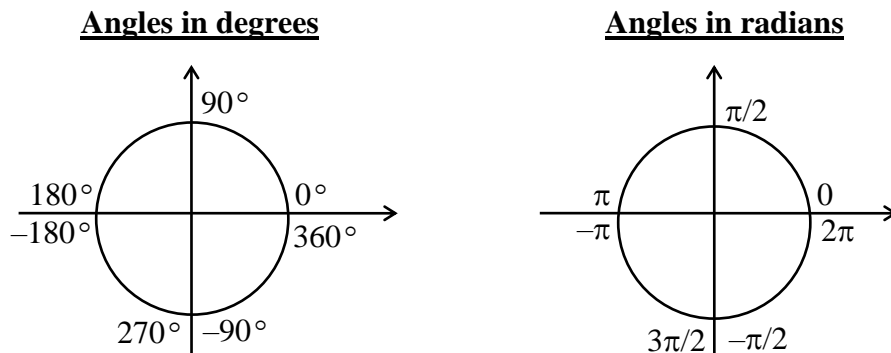
Exercises 19

In exercises 19.1 and 19.2, convert given angles from radians to degrees. Round the answers to the nearest tenth.

19.1. a) 1.3

    b) -0.6

19.2. a) 2.4

    b) -0.8

In exercises 19.3 and 19.4, convert given angles from radians to degrees.

19.3. a) $\dfrac{2\pi}{9}$

    b) $-\dfrac{3\pi}{10}$

19.4. a) $\dfrac{4\pi}{15}$

    b) $-\dfrac{6\pi}{5}$

In exercises 19.5 and 19.6, convert given angles from degrees to radians. Round the answers to the nearest hundreds.

19.5. a) $140°$

    b) $-85°$

19.6. a) $78°$

    b) $-237°$

In exercises 19.7 and 19.8, convert given angles from degrees to radians. Write the answers in terms of π. Do not round the answers.

19.7. a) $120°$

    b) $-150°$

19.8. a) $330°$

    b) $-225°$

In exercises 19.9 and 19.10, angle θ is given in radians. **Without** using a calculator, find the exact values of the sine, cosine, and tangent of the angle θ.

19.9. a) $\theta = \dfrac{4\pi}{3}$

    b) $\theta = -\dfrac{\pi}{4}$

    c) $\theta = \dfrac{5\pi}{6}$

19.10. a) $\theta = -\dfrac{\pi}{6}$

    b) $\theta = \dfrac{2\pi}{3}$

    c) $\theta = -\dfrac{3\pi}{4}$

In exercises 19.11 and 19.12, r is the radius of a circle, and θ is the central angle. Find the length of the arc bounded by angle θ. Round your answer to the nearest tenth.

19.11. a) $r = 4.2$ ft, $\theta = 2.3$ (rad)

    b) $r = 1.3$ cm, $\theta = 80°$

19.12. a) $r = 1.7$ m, $\theta = 0.3$ (rad)

    b) $r = 2.5$ in, $\theta = 115°$

Challenge Problems

19.13. The angles of a triangle are in the ratio of 3:4:5. Express these angles in radians.

19.14. The diameter of a Ferris wheel is 16 m. The spokes connecting two consecutive cabs to the center of the wheel make an angle of $\frac{\pi}{8}$. How many cabs are on the wheel? What is the length of the arc between two consecutive cabs?

19.15. Nick is running around a circular track of radius 30-meters. Esther is standing at the center and observing him. She found that she turned 6 radians in one minute. What was the Nick's speed?

19.16. A fly sat on the top of the second hand of a large clock and rode 48 cm until Lillian swatted it away. How long was the fly riding if the length of the second hand is 1.5 m?

19.17. In what quadrant does the 100-radian angle lie?

19.18. Which value is greater: sin 3 or cos 4? Here 3 and 4 are angles in radians. You may use a calculator only for simple arithmetic operations, but not for sine and cosine.

19.19. Prove the following statements:

a) Let n be any integer not divisible by 3. Then the reference angle of $\frac{n\pi}{3}$ is $\frac{\pi}{3}$.

b) Let n be any integer not divisible by 2. Then the reference angle of $\frac{n\pi}{4}$ is $\frac{\pi}{4}$.

c) Let n be any integer not divisible by 2 and 3. Then the reference angle of $\frac{n\pi}{6}$ is $\frac{\pi}{6}$.

d) Let k be any integer and $n = 3, 4, 5, \ldots$. Then the reference angle of $\frac{(kn+1)\pi}{n}$ is $\frac{\pi}{n}$.

Session 20

Graph and Basic Equation for Sine

In this and the next session, we will consider three basic trig functions: sine, cosine, and tangent. For these functions, we construct graphs, and solve the three basic (or simplest) equations: $\sin x = a$, $\cos x = a$, and $\tan x = a$, where a is a given constant number. We call these equations basic because the solution of many more complicated equations can be reduced to them. In this section, we focus on the sine, and in the next, on the cosine and tangent. We will use the radian measure.

Function $y = \sin(x)$

Let's recall the definition of the sine for an arbitrary angle: we draw the angle in the standard position in the system of coordinates with the unit circle, and consider the point of interception of the terminal side of the angle with the unit circle. Sine is the second (vertical) coordinate of this point.

To draw graph of sine, we will move along the unit circle, starting with the rightmost position, and observe how the vertical coordinate of the point on the unit circle changes from quadrant to quadrant.

Obviously, in the first quadrant, the vertical coordinate (i.e. sine) increases from zero to one:
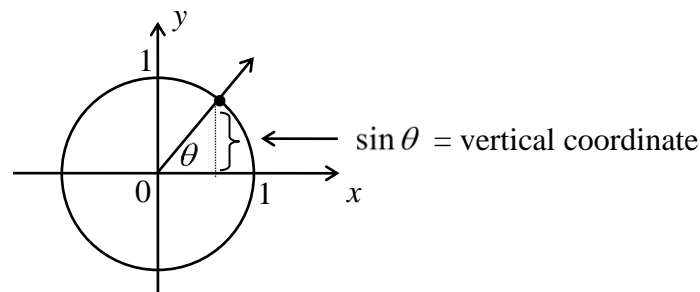
To graph the sine, we will use a different system of coordinates in which we mark the angle on the horizontal axis (we will use the letter x instead of θ), and mark $\sin x$ on the vertical y-axis. If you pick up several values of the angles x in the first quadrant (i.e. from 0 to $\pi/2$), calculate $\sin x$ (for instance, for special angles $\pi/6, \pi/4, \pi/3$), and plot the points in the system of coordinates, you will see that the sine does not increase along a straight line. Instead, it increases along the curve:
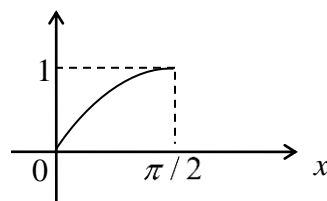
Graph of sine in the first quadrant

In similar way, in the second quadrant (from $\pi/2$ to π), sine decreases from 1 to 0:
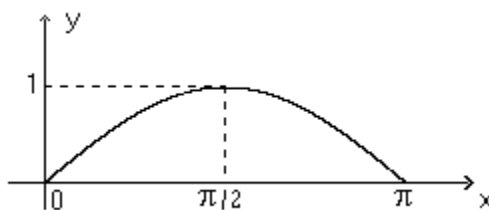
Graph of sine in the first and second quadrants

Continue moving along the unit circle, we see that in the third quadrant (from π to $3\pi/2$) sine decreases from 0 to –1, and in the fourth quadrant (from $3\pi/2$ to 2π) sine increases from –1 to 0. At this point, we get the graph of sine for one full cycle (we also say, on one period interval):
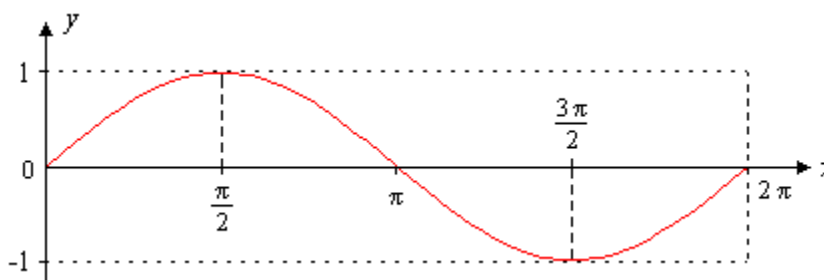
Graph of sine on one period interval $[0, \, 2\pi]$

If we continue to move around the unit circle in either direction (positive or negative), we will expand the graph of sine to the entire number line, i.e. for all the values of x from $-\infty$ to $+\infty$:
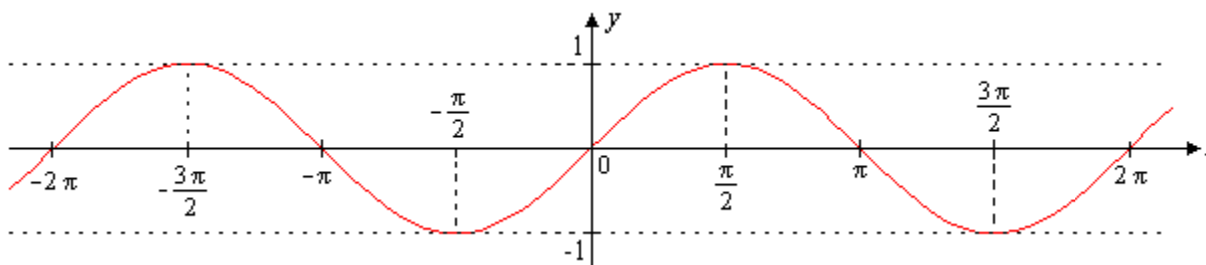
Entire graph of the function $y = \sin x$

You can see that the domain of sine (possible values of x) is the interval $(-\infty, +\infty)$ and the range (possible values of y) is $[-1, 1]$. Sine is a periodic function with a period of 2π. It means that the sine "repeats itself" on each interval of the length 2π. More formally, for any x

$$\sin(x + 2\pi) = \sin(x) \quad \text{(Periodic property of sine)}$$

Also, the graph is symmetric with respect to the origin. Algebraically, it means that

$$\sin(-x) = -\sin(x) \qquad \text{(Odd property of sine)}$$

Solving Basic Equation $\sin(x) = a$ on One Period Interval $[0,\, 2\pi)$

Notice that the right point 2π is not included in the interval $[0, 2\pi)$. The reason is that this point corresponds to the angle of 0, which is already taken for the left point of the interval.

In this interval, equation $\sin(x) = a$ may have zero, one or two solutions depending on the value of a. More precisely, the following statements are true.

Proposition 21.1. Consider the equation $\sin(x) = a$ in the interval $[0,\, 2\pi)$. Then

1) If $|a| > 1$, the equation does not have solutions.

2) If $|a| = 1$, the equation has one solution.

3) If $|a| < 1$, the equation has two solutions.

We can check all three statements using a geometric interpretation of the equation $\sin(x) = a$: we consider intersections of the horizontal line $y = a$ with the graph of sine. Then the x-coordinates of the points of intersections are the solutions. By drawing this horizontal line, we can see three different locations of this line, depending on three different values of number a.

1) $|a| > 1$. This inequality is equivalent to $a > 1$ or $a < -1$. Horizontal line $y = a$ is located above or below the graph of sine, so no points of intersection, and no solutions.

2) $|a| = 1$. This equality is equivalent to $a = 1$ or $a = -1$. In both cases line $y = a$ touches the graph of sine at only one point in the interval $[0,\, 2\pi)$:

    Line $y = 1$ touches the graph at the point $(\pi/2,\, 1)$. Therefore,
    the equation $\sin(x) = 1$ has only one solution $x = \pi/2$.

    Line $y = -1$ touches the graph at the point $(3\pi/2,\, -1)$. Therefore,
    the equation $\sin(x) = -1$ has only one solution $x = 3\pi/2$.

3) $|a| < 1$. This inequality is equivalent to $-1 < a < 1$. Line $y = a$ is located between the lines $y = -1$ and $y = 1$ and intersects the graph of sine at exactly two points in the interval $[0,\, 2\pi)$. In particular, if $a = 0$, equation $\sin(x) = 0$ has two solutions:

    $$x = 0 \quad \text{and} \quad x = \pi\,.$$

    To find the roots of the equation $\sin(x) = a$ when $a \ne 0$ and $-1 < a < 1$, we can use the reference angle. (For review of reference angles, see session 16).

Steps to solve the equation $\sin(x) = a,\ a \neq 0,\ -1 < a < 1.$

1) Set up the equation for the reference angle x_r: $\sin(x_r) = |a|$. In other words, ignore the sign of the number a (always take a plus sign). Solve this equation for the reference angle: $x_r = \sin^{-1}(|a|)$. To calculate x_r, you may use the button $\sin^{-1}$ on a calculator.

2) Determine the quadrants in which the angle x is located based on the sign of the number a:

   If $a > 0$, then angle x is located in the 1st and 2nd quadrants.

   If $a < 0$, then angle x is located in the 3rd and 4th quadrants.

3) Using quadrants and a reference angle, find two solutions x_1 and x_2 of the equation $\sin(x) = a$.

   If $a > 0$, then $x_1 = x_r$ and $x_2 = \pi - x_r$ (solutions are in the 1st and 2nd quadrants):
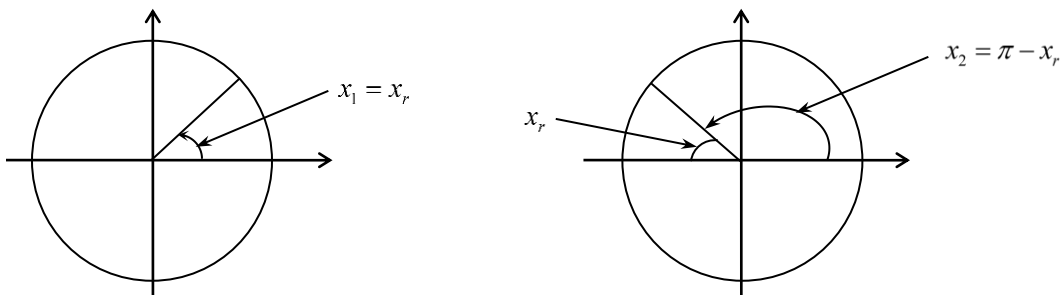
   If $a < 0$, then $x_1 = \pi + x_r$ and $x_2 = 2\pi - x_r$ (solutions are in the 3rd and 4th quadrants):
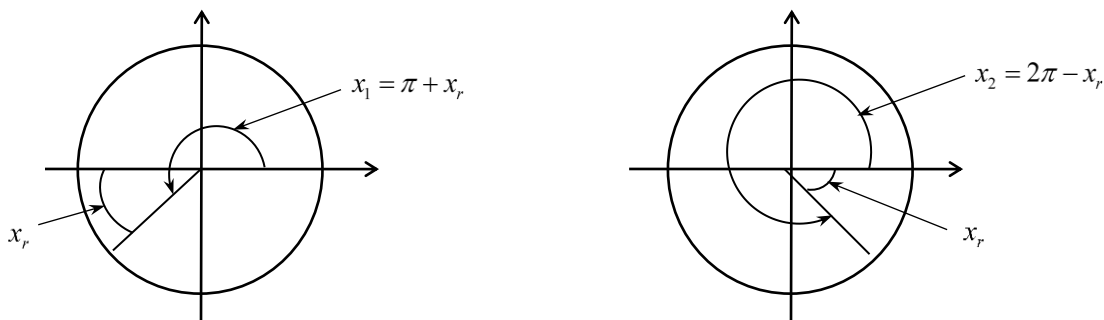
Example 20.1. Solve the equation $2\sin(x) + 4 = 5$ in the interval $[0,\ 2\pi)$.

Solution. We can reduce this equation to the basic one by solving for $\sin(x)$:

$$2\sin(x) = 1 \implies \sin(x) = \frac{1}{2}\ .$$

Here $a = 1/2 > 0$. Therefore, the equation for reference angle x_r is the same as for x: $\sin(x_r) = 1/2$. From here $x_r = \sin^{-1}(1/2)$. We can find x_r with a calculator or using the special value $1/2$: $x_r = \sin^{-1}(1/2) = 30° = \pi/6$. The original equation has two roots. One of them, x_1, is located in the 1st quadrant and coincides with the reference angle: $x_1 = x_r = \pi/6$. The second root x_2 is in the 2nd quadrant:

$$x_2 = \pi - x_r = \pi - \pi/6 = 5\pi/6.$$

Final answer: $\left\{ \dfrac{\pi}{6}, \dfrac{5\pi}{6} \right\}$.

Example 20.2. Solve the equation $-2\sin(x) = \sqrt{2}$ in the interval $[0, 2\pi)$.

Solution. Solving for $\sin(x)$, we get the basic equation $\sin(x) = -\sqrt{2}/2$. Here $a = -\sqrt{2}/2 < 0$. The equation for reference angle x_r is $\sin(x_r) = |a| = \sqrt{2}/2$. From here, using a calculator or the special value $\dfrac{\sqrt{2}}{2}$, we can find that $x_r = \sin^{-1}\left(\dfrac{\sqrt{2}}{2} \right) = 45° = \dfrac{\pi}{4}$.

The original equation has two roots. One of them, x_1, is located in the 3rd quadrant: $x_1 = \pi + x_r = \pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$. The second root x_2 is in the 4th quadrant: $x_2 = 2\pi - x_r = 2\pi - \dfrac{\pi}{4} = \dfrac{7\pi}{4}$.

Final answer: $\left\{ \dfrac{5\pi}{4}, \dfrac{7\pi}{4} \right\}$.

Example 20.3. Solve the equation $5\sin(x) - 1 = 3$ in the interval $[0, 2\pi)$. Round the answer to the nearest hundredth.

Solution. Solving for $\sin(x)$, we get the basic equation $\sin(x) = \dfrac{4}{5}$. Here $a = \dfrac{4}{5} > 0$. The equation for reference angle x_r is the same as for x: $\sin(x_r) = \dfrac{4}{5}$. The value $\dfrac{4}{5}$ is not a special value, and, to find x_r, use a calculator (make sure the calculator is in radian mode): $x_r = \sin^{-1}\left(\dfrac{4}{5} \right) = 0.93$ Original equation has two roots. One root x_1 is located in the 1st quadrant and coincides with the reference angle: $x_1 = x_r = 0.93$ The second root x_2 is in the 2nd quadrant: $x_2 = \pi - x_r = \pi - 0.93 = 2.21$.

Final answer: $\{0.93, 2.21\}$.

Example 20.4. Solve the equation $6\sin(x)+7=2$ in the interval $[0,\,2\pi)$. Round the answer to the nearest hundredth.

Solution. Solving for $\sin(x)$, we get the basic equation $\sin(x)=-\dfrac{5}{6}$. Here $a=-\dfrac{5}{6}<0$.

The equation for reference angle x_r is $\sin(x_r)=\left|-\dfrac{5}{6}\right|=\dfrac{5}{6}$. The value $\dfrac{5}{6}$ is not a special value and, to find x_r, we use a calculator in the radian mode: $x_r=\sin^{-1}\left(\dfrac{5}{6}\right)=0.99$. The original equation has two roots. One of them, x_1, is located in the 3rd quadrant: $x_1=\pi+x_r=\pi+0.99=4.13$. The second root x_2 is in the 4th quadrant: $x_2=2\pi-x_r=2\pi-0.99=5.29$.

Final answer: $\{4.13,\,5.29\}$.

Exercises 20

In all exercises and problems, solve the given equations in the interval $[0, 2\pi)$. Use the radian measure.

In exercises 20.1 – 20.2, do not round the answers. Write the answers in terms of π.

20.1. a) $2\sqrt{2}\sin(x) - 1 = 1$

    b) $2\sqrt{3}\sin(x) + 4 = 1$

    c) $2\sqrt{3}\sin(x) - 3 = 1$

20.2. a) $2\sqrt{3}\sin(x) - 1 = 2$

    b) $2\sqrt{2}\sin(x) + 3 = 1$

    c) $3\sqrt{2}\sin(x) - 4 = 2$

In exercises 20.3 and 20.4, the equations are not linear. However, they can be reduced to the linear equations by treating them as quadratic equations with respect to the corresponding trig functions. See more non-linear equations in session 22.

20.3. a) $2\sin^2 x + \sin x = 0$

    b) $2\sin^2 x + \sin x - 1 = 0$

20.4. a) $2\sin^2 x - \sin x = 0$

    b) $2\sin^2 x - \sin x - 1 = 0$

In exercises 20.5 and 20.6, round the answers to the nearest hundredth.

20.5. a) $4\sin(x) - 1 = 2$

    b) $6\sin(x) + 5 = 3$

20.6. a) $5\sin(x) - 2 = 1$

    b) $7\sin(x) + 6 = 2$

Challenge Problems

20.7. $\sin^3 x + \sin x = 2$

20.8. $\sin^4 x - \sin x = 2$

Session 21

Graphs and Basic Equations for
Cosine and Tangent

In the previous session, we examined the graph and the basic equation for the sine function. Here we will study these topics for cosine and tangent. Namely, we will construct graphs for cosine and tangent, and solve the basic equations $\cos x = a$ and $\tan x = a$. As in the previous session, we will use the radian measure.

Function $y = \cos(x)$

We can proceed similar to the sine function. We will do this briefly, hoping that the reader will be able to restore the details on their own. By definition, cosine is the first (horizontal) coordinate of a point on the unit circle that corresponds to given angle θ:
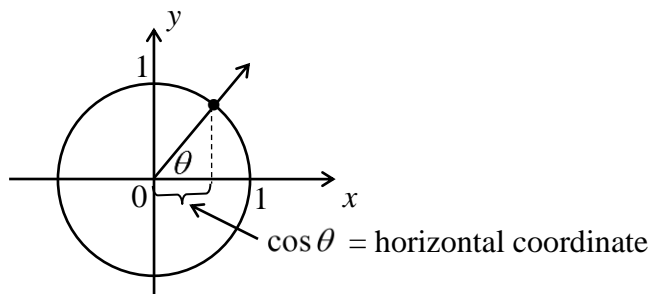
Moving around the unit circle from quadrant to quadrant, we can plot the graph of cosine by observing how the horizontal coordinate changes. As in the case of the sine function, we use another coordinate system in which we mark the angle on the horizontal axis (we will use the letter x instead of θ), and mark $\cos x$ on the vertical y-axis. In the first quadrant, when the angle runs from 0 to $\pi/2$, cosine decreases from 1 to 0:
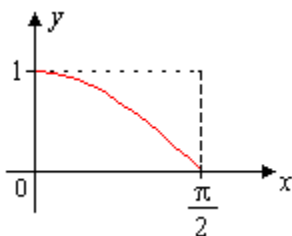
Graph of the cosine in the first quadrant

In the second quadrant, cosine continues to decrease from 0 to –1, in the third quadrant, it increases from –1 to 0, and, finally, in the fourth quadrant, it increases from 0 to 1. Here is the graph of the cosine at one full cycle (on one period interval) from 0 to 2π:
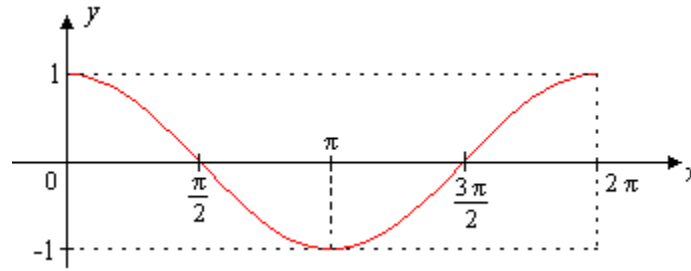
Graph of the cosine on one period interval $[0, 2\pi]$

If we extend graph to the entire x-axis, we get the complete graph of the cosine:
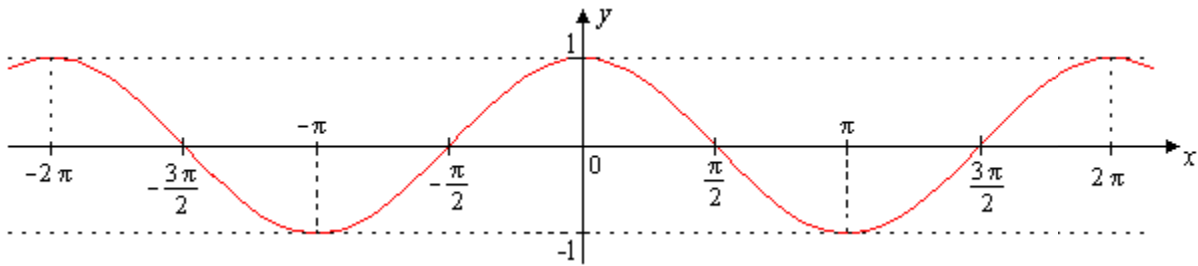
Entire graph of the function $y = \cos x$

Like for the sine, the domain of the cosine is $(-\infty, +\infty)$, range is $[-1, 1]$, and the cosine is a periodical function with the same period 2π. Graph of the cosine is symmetric about the y-axis:

$$\cos(-x) = \cos(x) \text{ (Even property of cosine).}$$

Note. It is also possible to get the graph of the cosine by shifting the graph of sine to the left by $\pi/2$ using the reduction formula $\cos(x) = \sin(x + \pi/2)$.

Solving Basic Equation $\cos(x) = a$ on One Period Interval $[0, 2\pi)$

The number of solutions of this equation is exactly the same as that of the sine.

Proposition 21.1. Consider the equation $\cos(x) = a$ on the interval $[0, 2\pi)$. Then

1) If $|a| > 1$, the equation does not have solutions.

2) If $|a| = 1$, the equation has one solution in the interval $[0, 2\pi)$:

   For the equation $\cos(x) = 1$, the only solution is $x = 0$.

   For the equation $\cos(x) = -1$, the only solution is $x = \pi$.

3) If $|a| < 1$, the equation has two roots. In particular, if $a = 0$, roots of the equation $\cos(x) = 0$ are $x_1 = \pi/2$ and $x_2 = 3\pi/2$. If $a \neq 0$, both roots can be found using

the reference angle $x_r = \cos^{-1}(|a|)$ in the same way as we did for the equation $\sin(x) = a$. Also, the following formulas can be used that are true for any values of a, such that $|a| < 1$:

$$x_1 = \cos^{-1}(a) \text{ and } x_2 = 2\pi - \cos^{-1}(a).$$

Note. We described two methods of solving the equation $\cos(x) = a$: use steps similar to those described for the equation $\sin(x) = a$, and the above formulas.

Example 21.1. Solve the equation $2\sqrt{3}\cos(x) - 1 = 2$ in the interval $[0, 2\pi)$.

Solution. First, to get the basic equation, we solve this equation for $\cos(x)$. We have $\cos(x) = \dfrac{3}{2\sqrt{3}}$ or $\cos(x) = \dfrac{\sqrt{3}}{2}$. Using steps similar to equation $\sin(x) = a$, we set up the equation for the reference angle x_r and solve it:

$$\cos(x_r) = \left|\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2} \Rightarrow x_r = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30° = \frac{\pi}{6}.$$

Since $\cos(x) > 0$, given equation has two roots which are located in the 1st and 4th quadrants:

$$x_1 = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30° = \frac{\pi}{6}, \text{ and } x_2 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$$

Example 21.2. Solve the equation $2\cos(x) + 4 = 3$ in the interval $[0, 2\pi)$.

Solution. Solving this equation for $\cos(x)$, we have $\cos(x) = -\dfrac{1}{2}$. Using the second method indicated in the Note above, we get two solutions:

$$x_1 = \cos^{-1}\left(-\frac{1}{2}\right) = 120° = \frac{2\pi}{3}, \text{ and } x_2 = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}.$$

Function $y = \tan(x)$

On the unit circle in the system of coordinates, we can interpret tangent as follows. On the right side of the unit circle, draw a vertical line and extend the terminal side of the angle to meet with that vertical line. Then the tangent is the vertical coordinate of the point of interception. Here are pictures of the tangent when the angle is located in each of the quadrants:
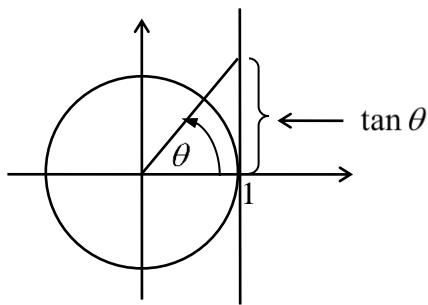
Angle is in 1st quadrant   Angle is in 2nd quadrant
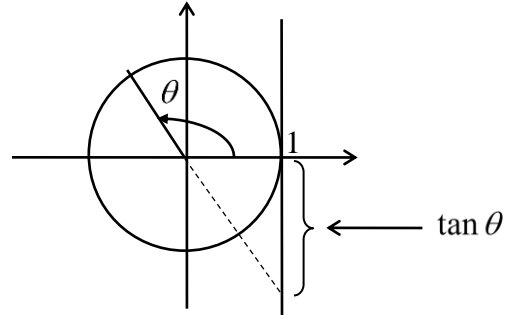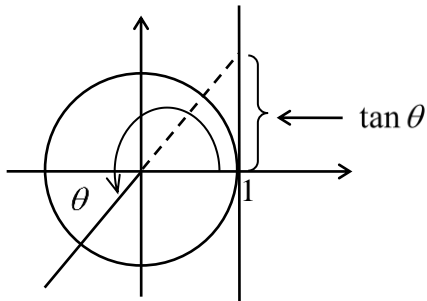
Angle is in 3rd quadrant   Angle is in 4th quadrant
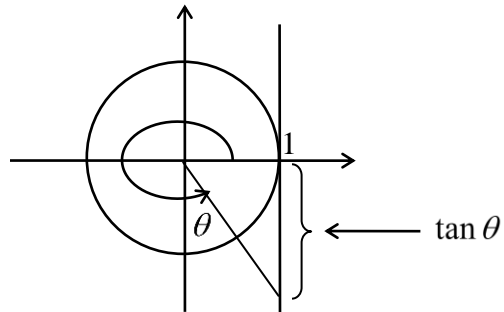
We will draw the graph of the tangent in the same way as we did for the sine and cosine. Moving along the unit circle in the first quadrant, note that the tangent increases from zero to infinity, and its graph in the 1st quadrant is as follows:
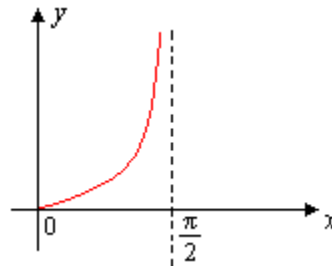
Graph of the tangent in the first quadrant

Line $x = \pi/2$ becomes vertical asymptote. Continue moving in 2nd quadrant, we get the picture
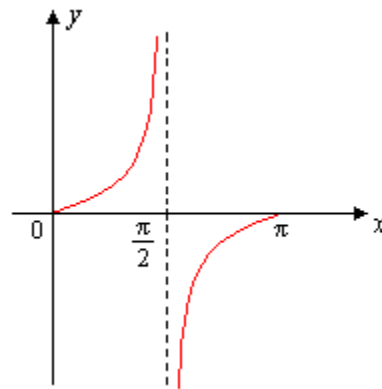
Graph of the tangent in the first and second quadrants

Moving in the 3rd and 4th quadrants, we get graph of tangent on interval $[0, 2\pi)$:
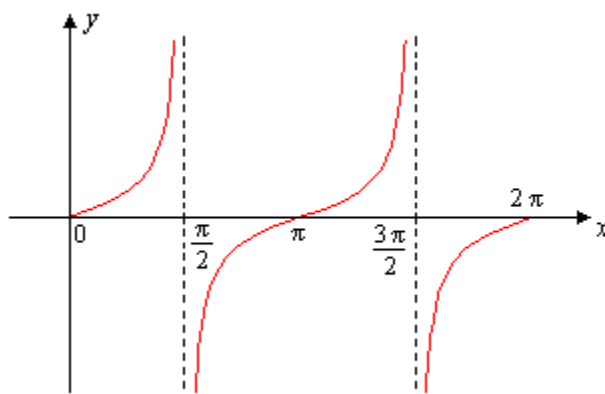
Graph of tangent on the interval $[0, 2\pi]$

Continue to move around the unit circle in both directions, we can draw a complete graph of the tangent:
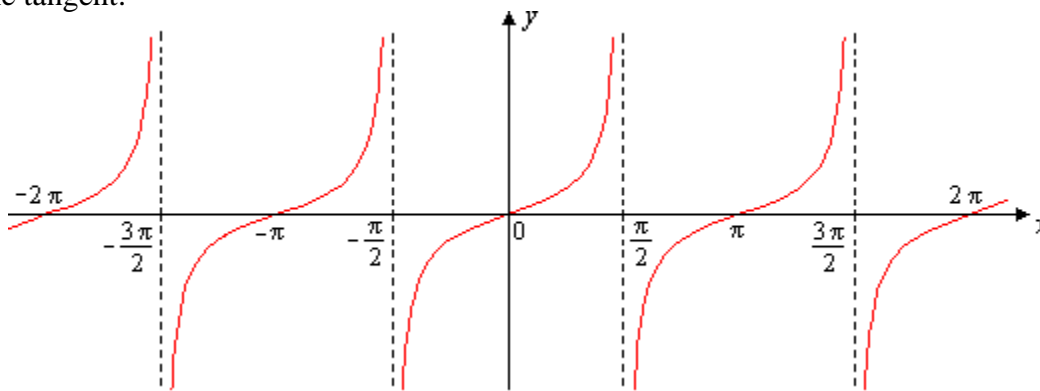
Entire graph of the function $y = \tan x$

We see that the graph consists of an infinite number of branches and has an infinite number of vertical asymptotes. The graph is symmetrical about the origin, so the tangent is an odd function: $\tan(-x) = -\tan(x)$. It repeats over an interval of length π, so the tangent has a period of π: $\tan(x+\pi) = \tan(x)$.

Solving the Basic Equation $\tan(x) = a$ on the Interval $[0, 2\pi)$

Any horizontal line $y = a$ intersects the graph of a tangent on $[0, 2\pi)$ interval always at two points (see pictures above), so on this interval the equation $\tan(x) = a$ always has exactly two roots for any a. The roots can be found in the same way as we described in session 20 for the equation $\sin(x) = a$, using the reference angle x_r which is a solution of the equation $\tan(x_r) = |a|$: $x_r = \tan^{-1}(|a|)$. To calculate the reference angle, we can use

the button $\tan^{-1}$ on the calculator.

Proposition 21.2. For any a, the equation $\tan(x) = a$ has two roots in the interval $[0, 2\pi)$.

If $a = 0$, the roots are $x_1 = 0, \ x_2 = \pi$.

If $a \neq 0$, then using the reference angle $x_r = \tan^{-1}(|a|)$, we have

   1) If $a > 0$, the roots are in the 1st and 3rd quadrants and are
$$x_1 = x_r, \ x_2 = \pi + x_r.$$

   2) If $a < 0$, the roots are in the 2nd and 4th quadrants and are
$$x_1 = \pi - x_r, \ x_2 = 2\pi - x_r.$$

Note. You do not need to memorize these formulas. Just draw a unit circle and mark the corresponding angles, as we did for the equation $\sin(x) = a$.

Example 21.3. Solve the equation $3\tan(x) - 2\sqrt{3} = \sqrt{3}$ in the interval $[0, 2\pi)$.

Solution. Solving the equation for $\tan(x)$, we get the basic equation $\tan(x) = \sqrt{3}$. The reference angle x_r is the solution of the equation $\tan(x_r) = |\sqrt{3}| = \sqrt{3}$:
$$x_r = \tan^{-1}(\sqrt{3}) = 60° = \pi/3.$$

Roots of the given equation are located in the 1st and 3rd quadrants and are
$$x_1 = x_r = \pi/3,$$
$$x_2 = \pi + x_r = \pi + \pi/3 = 4\pi/3.$$

Example 21.4. Solve the equation $4\tan(x) + 5 = 1$ in the interval $[0, 2\pi)$.

Solution. Solving the equation for $\tan(x)$, we obtain the basic equation $\tan(x) = -1$. The reference angle x_r is the solution of the equation $\tan(x_r) = |-1| = 1$:
$$x_r = \tan^{-1}(1) = 45° = \pi/4.$$

Roots of the given equation are located in the 2nd and 4th quadrants and are
$$x_1 = \pi - x_r = \pi - \pi/4 = 3\pi/4,$$
$$x_2 = 2\pi - x_r = 2\pi - \pi/4 = 7\pi/4.$$

Exercises 21

In all exercises, solve the given equations in the interval $[0, 2\pi)$. Use a radian measure.

In exercises 21.1 – 21.4, do not round the answers. Write the answers in terms of π.

21.1. a) $2\sqrt{2}\cos(x) + 3 = 5$

    b) $2\sqrt{3}\cos(x) + 5 = 2$

    c) $3\sqrt{2}\cos(x) - 7 = 2$

21.2. a) $2\sqrt{3}\cos(x) + 3 = 6$

    b) $\sqrt{2}\cos(x) + 4 = 3$

    c) $2\sqrt{3}\cos(x) - 5 = 1$

21.3. a) $\sqrt{3}\tan(x) + 2 = 3$

    b) $\sqrt{3}\tan(x) + 6 = 3$

21.4. a) $\sqrt{3}\tan(x) + 2 = 5$

    b) $\sqrt{3}\tan(x) + 4 = 3$

In exercises 21.5 and 21.6, the equations are not linear. However, they can be reduced to the linear equations by treating them as quadratic equations with respect to the corresponding trig functions. See more non-linear equations in the next session.

21.5 a) $2\cos^2 x - \cos x = 0$

    b) $2\cos^2 x - \cos x - 1 = 0$

    c) $\tan^2 x - \tan x = 0$

21.6. a) $2\cos^2 x + \cos x = 0$

    b) $2\cos^2 x + \cos x - 1 = 0$

    c) $\tan^2 x + \tan x = 0$

In exercises 21.7 – 21.10, round the answers to the nearest hundredth.

21.7. a) $5\cos(x) - 2 = 1$

    b) $7\cos(x) + 6 = 2$

21.8. a) $4\cos(x) - 1 = 2$

    b) $6\cos(x) + 5 = 3$

21.9. a) $3\tan(x) - 1 = 5$

    b) $4\tan(x) + 7 = 5$

21.10. a) $2\tan(x) - 3 = 5$

    b) $3\tan(x) + 4 = 2$

Challenge Problems

21.11. Prove that the following equations have solutions in the interval $[0, 2\pi)$ only for the indicated values of a. Find the corresponding solutions. The letter n below means any integer.

    a) $\sin(x) + \cos(ax) = 2$, $a = 4n$.

    b) $\sin(x) - \cos(ax) = 2$, $a = 4n + 2$.

    c) $\sin(ax)\cos(x) = 1$, $a = (4n - 1)/2$.

    d) $\sin(x)\cos(ax) = 1$, $a = 4n$, or $a = (4n + 2)/3$.

Session 22

Trigonometric Identities and None-Basic Equations

Trigonometric Identities

Let's compare the two statements:

$$\sin x + \cos x = 1 \text{ and } \sin^2 x + \cos^2 x = 1.$$

They look pretty much similar. However, they are completely different. The first one is the **equation** and the second one is the **identity**.

As we already know, equation is a statement that is true only for some specific values of variable, and the main problem for equation is to **solve** this, which means to find these specific values. Such values are called solutions or roots of the equation. For example, the values $x = 0$ and $x = \pi / 2$ are roots of the equation $\sin x + \cos x = 1$. We can check this by substituting these values in the equation. We will have $\sin 0 + \cos 0 = 0 + 1 = 1$ and $\sin(\pi/2) + \cos(\pi/2) = 1 + 0 = 1$, so the equation becomes a true statement. If we pick, for example, $x = \pi$, it is not a root, because by substituting π for x, the equation does not become a true statement: $\sin(\pi) + \cos(\pi) = 0 - 1 = -1 \neq 1$. In example 22.7 below, we show that roots 0 and $\pi / 2$ are the only roots of the equation $\sin x + \cos x = 1$ in the interval $[0, \, 2\pi)$.

On the contrary, the second statement $\sin^2 x + \cos^2 x = 1$ is true for any value of x, no exceptions. We can check this using the definitions of sine and cosine as the vertical and horizontal coordinates of points on the unit circle:
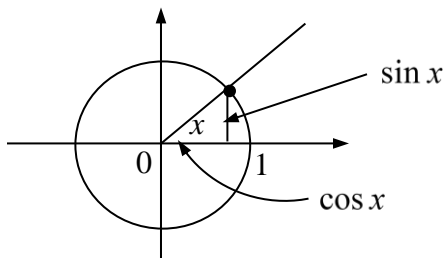
Using these coordinates (i.e. $\sin x$ and $\cos x$) together with the radius (which is equal to 1) we can form a right triangle if a point $(\sin x, \, \cos x)$ lies in the 1$^{\text{st}}$ quadrant. By the Pythagorean Theorem, $\sin^2 x + \cos^2 x = 1$. We can check that this statement is true for all quadrants. Statements like this are called identities. Here is the exact definition.

Definition. Statement $f(x) = g(x)$, where f and g are two functions, is called the **identity**, if this statement is true for all values of variable x from the common domain of functions f and g.

The main problem for identity is to **prove** it, not to solve. In general, it is not possible to give an exact recipe how to prove an identity. Common guideline is to try to modify one or both sides f and g of the identity to get the same expression. Below we consider several examples.

Note: formally speaking, we can treat the identity $f = g$ as a special case of an equation, that has solutions for any point from the common domain of the functions f and g. However, here we distinguish between the identity and the equation.

Let's start with identities that we call **basic**. They can be used to prove more complicated identities. Four of them are simply expressions of $\tan x$, $\cot x$, $\sec x$ and $\csc x$ through $\sin x$ and $\cos x$:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

Another identity that we have already proven above is:

$$\sin^2 x + \cos^2 x = 1$$

We call this the **main** identity (it is also called the Pythagorean identity). It allows us to express $\sin^2 x$ through $\cos^2 x$ and vice versa:

$$\sin^2 x = 1 - \cos^2 x \quad \text{and} \quad \cos^2 x = 1 - \sin^2 x.$$

From the main identity, we can derive two more identities that connect $\tan x$ with $\sec x$, and $\cot x$ with $\csc x$: just divide both sides of the main identity by $\cos^2 x$ and $\sin^2 x$:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{and} \quad \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

From here we get

$$\tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$

There are a lot of trig identities that can be derived from the basics. Let's consider some examples.

Example 22.1. Prove the identity $\left(\tan x + 1\right)^2 = 2\tan x + \sec^2 x$.

Solution. $\left(\tan x + 1\right)^2 = \tan^2 x + 2\tan x + 1 = \left(\tan^2 x + 1\right) + 2\tan x = \sec^2 x + 2\tan x$.

Example 22.2. Prove the identity $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \sec^2 x \cdot \csc^2 x$.

Solution. We will modify the left side to get the right side:

$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cdot \cos^2 x} = \frac{1}{\sin^2 x \cdot \cos^2 x} = \frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = \csc^2 x \cdot \sec^2 x \, .$$

Example 22.3. Prove the identity $\sin^2 x - \cos^2 x = \sin^4 x - \cos^4 x$.

Solution. This time we modify the right side using the formula $a^2 - b^2 = (a-b) \cdot (a+b)$:

$$\sin^4 x - \cos^4 x = \left(\sin^2 x - \cos^2 x\right) \cdot \left(\sin^2 x + \cos^2 x\right) = \sin^2 x - \cos^2 x \, .$$

Example 22.4. Prove the identity $\dfrac{1 + \sin x}{1 - \sin x} = \left(\tan x + \sec x\right)^2$.

Solution. Here both sides look quite complicated, and we modify both of them.

To modify the left side, we multiply the numerator and denominator by $1 + \sin x$. Then, using the identity $1 - \sin^2 x = \cos^2 x$ for the denominator, we get

$$\frac{1 + \sin x}{1 - \sin x} = \frac{(1 + \sin x)(1 + \sin x)}{(1 - \sin x)(1 + \sin x)} = \frac{(1 + \sin x)^2}{1 - \sin^2 x} = \frac{1 + 2\sin x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x + 2\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} + \tan^2 x$$

$$= \sec^2 x + 2\tan x \cdot \sec x + \tan^2 x.$$

Now let's modify the right side of the original identity:

$$\left(\tan x + \sec x\right)^2 = \tan^2 x + 2\tan x \cdot \sec x + \sec^2 x$$

We've got the same expression as for the left side. Identity is proved.

None-Basic Trigonometric Equations

In two previous sessions we solved basic trig equations $\sin x = a$, $\cos x = a$ and $\tan x = a$. Now we consider slightly more complicated equations that can be reduced to basic. To solve some of the equations, we will use basic identities. All equations we will solve in radians and in the interval $[0, 2\pi)$.

Example 22.5. Solve the equation $8\sin^2 x + 14\sin x - 15 = 0$. Round the answer to the nearest hundredth.

Solution. This equation can be treated as a quadratic equation with respect to $\sin x$. Using the notation $\sin x = u$, the equation becomes quadratic with respect to variable u: $8u^2 + 14u - 15 = 0$. It can be solved by the method described in session 2:

1) Construct the reduced equation by multiplying the last coefficient –15 by the leading coefficient 8: $u^2 + 14u - 15 \cdot 8 = 0$ or $u^2 - 14u - 120 = 0$.
2) Solve the above reduced equation: $(u - 6)(u + 20) = 0 \Rightarrow u = 6$ and $u = -20$.

3) Get the roots of the original equation by dividing 6 and –20 by the leading coefficient 8:

$$u = \frac{6}{8} = \frac{3}{4} = 0.75 \text{ and } u = \frac{-20}{8} = -\frac{5}{2} = -2.5.$$

Replacing u with $\sin x$, we get two basic trig equations: $\sin x = 0.75$ and $\sin x = -2.5$. We can solve them in the same way as we did in the previous sessions.

1) Equation $\sin x = 0.75$ has two solutions in the interval $[0, 2\pi)$:
$$x = \sin^{-1}(0.75) = 0.85 \text{ and } x = \pi - \sin^{-1}(0.75) = 2.29.$$

2) Equation $\sin x = -2.5$ does not have solutions because $\sin x$ cannot be less than -1.

Final answer: there are two solutions $x = 0.85$ and $x = 2.29$.

Example 22.6. Solve the equation $8\sin^2 x - 2\cos x - 5 = 0$.

Solution. We can rewrite this equation in terms of $\cos x$ using the identity $\sin^2 x = 1 - \cos^2 x$. Substituting this expression into the original equation, we obtain

$$8(1 - \cos^2 x) + 2\cos x - 5 = 0 \Rightarrow 8 - 8\cos^2 x - 2\cos x - 5 = 0 \Rightarrow$$
$$-8\cos^2 x - 2\cos x + 3 = 0 \Rightarrow 8\cos^2 x + 2\cos x - 3 = 0.$$

Similar to example 22.5, we can treat this equation as a quadratic with respect to $\cos x$. Letting $u = \cos x$, we have the equation $8u^2 + 2u - 3 = 0$. It can be solved in the same way as in example 22.5. We omit the details and get $u = \frac{1}{2} = 0.5$ and $u = -\frac{3}{4} = -0.75$.

Replacing u with $\cos x$, we get two basic trig equation $\cos x = 0.5$ and $\cos x = -0.75$. Let's solve them.

1) Equation $\cos x = 0.5$ has two solutions (which are special angles)

$$x = \cos^{-1}(0.5) = 60° = \frac{\pi}{3} \text{ and } x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

2) Equation $\cos x = -0.75$ also has two solutions (which can be find using the calculator). We round the answer to the nearest hundredth:

$$x = \cos^{-1}(-0.75) = 2.42 \text{ and } x = 2\pi - \cos^{-1}(-0.75) = 3.86.$$

Final answer: there are four roots: $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$, $x = 2.42$, and $x = 3.86$.

Note. The first two roots are exact solutions, while the last two are approximations to two decimal places.

Example 22.7. Solve the equation $2\sin x \cdot \tan x + \sqrt{3} \tan x = 0$.

Solution. We can factor out $\tan x$: $\tan x\left(2\sin x+\sqrt{3}\right)=0$. Now equation can be split into two: $\tan x=0$ and $2\sin x+\sqrt{3}=0$. The first one is basic, and the second one can be written as basic: $\sin x=-\sqrt{3}/2$. Let's solve them. Both of them have two solutions. For $\tan x=0$, $x=0$ and $x=\pi$.

For $\sin x=-\sqrt{3}/2$ we can use steps described in Session 20:
Consider the equation for the reference angle x_r:

$$\sin x_r =\left|-\sqrt{3}/2\right|=\sqrt{3}/2 \;\Rightarrow\; x_r=60°=\pi/3.$$

Since $\sin x<0$, angle x is located in the 3rd and 4th quadrants, and we have two solutions:

$$x=\pi+x_r=\pi+\pi/3=4\pi/3 \text{ and } x=2\pi-x_r=2\pi-\pi/3=5\pi/3.$$

Final answer: there are four solutions: $x=0,\; x=\pi,\; x=4\pi/3,\; x=5\pi/3$.

Example 22.8. Solve the equation $\sin x+\cos x=1$.
Solution. Let's square both sides:

$$\left(\sin x+\cos x\right)^2 =1^2 \Rightarrow \sin^2 x+2\sin x\cdot\cos x+\cos^2 x=1.$$

Using the main identity $\sin^2 x+\cos^2 x=1$, we can simplify the above equation:

$$1+2\sin x\cdot\cos x=1 \;\Rightarrow\; 2\sin x\cdot\cos x=0 \;\Rightarrow\; \sin x\cdot\cos x=0 .$$

This equation can be split into two basic equations: $\sin x=0$ and $\cos x=0$.
Equation $\sin x=0$ has two solutions $x=0$ and $x=\pi$. Equation $\cos x=0$ also has two solutions $x=\dfrac{\pi}{2}$ and $x=2\pi-\dfrac{\pi}{2}=\dfrac{3\pi}{2}$. So, it looks like the original equation has four solutions: $x=0,\; x=\pi,\; x=\dfrac{\pi}{2},\; x=\dfrac{3\pi}{2}$. However, **this is not true**. We need to be very careful and check these values with the original equation. The reason is that when we square both sides of an equation, we can get additional roots that are extraneous to the original equation. We already saw that in session 8, examples 8.6 and 8.8. Let's check the above four values.

The values $x=0$, $x=\dfrac{\pi}{2}$ and $x=\pi$ we already checked at the beginning of this session:

0 and $\dfrac{\pi}{2}$ are roots, but $x=\pi$ is not. Let's check $x=\dfrac{3\pi}{2}$:

$$\sin\left(\dfrac{3\pi}{2}\right)+\cos\left(\dfrac{3\pi}{2}\right)=-1+0=-1\neq1. \text{ So, } x=\dfrac{3\pi}{2} \text{ is not a root, and we reject it.}$$

Final answer: there are only two roots $x=0$ and $x=\dfrac{\pi}{2}$.

Exercises 22

In exercises 22.1 – 22.12, prove the given identities.

22.1. $\cos^2 x \cdot \tan^2 x = 1 - \cos^2 x$

22.2. $\sin^2 x \cdot \cot^2 x = 1 - \sin^2 x$

22.3. $\cot x \cdot \cos x = \csc x - \sin x$

22.4. $\tan x \cdot \sin x = \sec x - \cos x$

22.5. $\csc x - \sin x = \dfrac{\cos x}{\tan x}$

22.6. $\sec x - \cos x = \dfrac{\sin x}{\cot x}$

22.7. $\dfrac{\tan x}{1 - \cos x} - \dfrac{\tan x}{1 + \cos x} = 2\csc x$

22.8. $\dfrac{\cot x}{1 - \sin x} - \dfrac{\cot x}{1 + \sin x} = 2\sec x$

22.9. $\dfrac{\cot x}{1 + \csc x} - \dfrac{\cot x}{1 - \csc x} = 2\sec x$

22.10. $\dfrac{\tan x}{1 + \sec x} - \dfrac{\tan x}{1 - \sec x} = 2\csc x$

22.11. $\dfrac{\cos^4 x - \sin^4 x}{\cos^2 x} = 2 - \sec^2 x$

22.12. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x} = 2 - \csc^2 x$

In exercises 22.13 – 22.26, solve the given equations in the interval $[0, 2\pi)$. Use radian measure. If it is possible to find exact solutions, write them in terms of π. If it is not possible, round the answers to the nearest thousandth.

22.13. $\cos^2(x) = \cos(x)$

22.14. $\sin^2(x) = \sin(x)$

22.15. $\sin^2(x) + \sin(x) = 0$

22.16. $\cos^2(x) + \cos(x) = 0$

22.17. $4\cos^2(x) - 1 = 1$

22.18. $6\sin^2(x) + 1 = 4$

22.19. $2\sin^2(x) - 3\sin(x) = 2$

22.20. $2\cos^2(x) + 7\cos(x) = -3$

22.21. $3\sin^2(x) - \cos(x) - 1 = 0$

22.22. $5\cos^2(x) + \sin(x) - 1 = 0$

22.23. $3\sin(x) \cdot \tan(x) = \sqrt{3}\sin(x)$

22.24. $\cos(x) \cdot \tan(x) = \sqrt{3}\cos(x)$

22.25. $\sin(x) - \cos(x) = 1$

22.26. $\cos(x) - \sin(x) = 1$

Challenge Problem

22.27. Solve the following equation in the interval $[0, 2\pi)$:

$$81^{\sin^2 x} + 81^{\cos^2 x} = 30$$

Hint: There are eight roots. To find them, you can use a new variable $u = 81^{\sin^2 x}$.

Part III

Exponential and Logarithmic Functions

Session 23

Logarithms

Consider the following problem. Suppose we have three numbers x, y and z, that are connected by the equation $y^x = z$. How to solve this equation for x and for y?

It is easy to solve for y: raise both sides of the equation to the power $\dfrac{1}{x}$, and get

$$\left(y^x\right)^{\frac{1}{x}} = y^{x \cdot \frac{1}{x}} = y^{\frac{x}{x}} = y^1 = y = z^{\frac{1}{x}}. \text{ So, } y = z^{\frac{1}{x}}.$$

It is important to understand that even we obtained the formula $y = z^{\frac{1}{x}}$, this formula, in general, does not give us a direct way (a finite sequence of arithmetic operations) to get the exact answer. Actually, the expression $z^{\frac{1}{x}}$ only provides a **notation** for a specific operation on x and z, and the question of how to perform this operation is a separate story (which is beyond the scope of this textbook).

Similar situation occurs when we want to solve the equation $y^x = z$ for x. In other words, we want to express power x in terms of the base y and number z. Of course, for some specific values of y and z, it is easy to do.

Example 23.1. Solve the equation $2^x = 8$.

Solution. This equation can be solved directly. Indeed, we can represent the number 8 in the exponential form with the base of 2: $8 = 2^3$. Then the equation takes the form $2^x = 2^3$. From here we immediately conclude that $x = 3$.

However, in general, we cannot solve the equation $y^x = z$ for x so easily. Consider, for example, equation $2^x = 6$. Because $2^2 = 4$ and $2^3 = 8$, we can just estimate that x should be somewhere between 2 and 3. But where? We cannot indicate the exact value. At the end of this session we will be able to get an approximation. We will see in Example 23.8 that up to three decimals, $x \approx 2.585$.

In general, we may think of the solution x of the equation $y^x = z$ as a result of some specific operation that we perform on y and z. In other words, we consider x as some function of two variables y and z. Since we have a function, we need a notation for it. You may invent your own notation. For example, using the abbreviation "sol" for solution, we can write $x = \text{sol}(y, z)$. In mathematics, the following notation is used: $x = \log_y z$. We read this as the "logarithm (or, in short, log) of the number z with the base y". So, the solution of the equation $y^x = z$ with respect to x is $x = \log_y z$.

In the definition below, we simply change letter y to b and z to c.

Definition of Logarithm. Let b be a positive number not equal to 1, and c be any positive number. Then $x = \log_b c$ is the solution of the equation $b^x = c$. In other words, the logarithm is a **power** to which we raise the base b to get the number c.

Note. You may be wondering why the base $b \neq 1$. Well, let's $b = 1$, so we consider the equation $1^x = c$. If we raise 1 to any power, the result is still 1, so we have $c = 1$, and any number x satisfies the equation $1^x = 1$. Therefore, in this case, the solution $x = \log_1 1$ does not make sense, since it can be any number. Another restriction is $b > 0$. This is to avoid problems with complex (not real) numbers. For example, $(-1)^{1/2} = \sqrt{-1}$ is not a real number, so we exclude negative base b. Also, we put the restriction on number c: $c > 0$. This is because $c = b^x$, and a positive b raised to any power is positive, so for a non-positive c logarithm does not exist.

In practice, often number 10 is used as the base of logarithms. Such logarithms are called **common** ones. Usually, for simplicity we drop the base 10 in the notation of common logs. So,

$$\log c = \log_{10} c$$

When working with logs, it is often convenient to convert them into exponential expressions. If we denote given logarithm by x (i.e. $\log_b c = x$), we can re-write it (by definition) as $b^x = c$. To make it easier to use this technique, you can memorize the following **"Circular Rule"** for conversion: in $\log_b c = x$, take the base b, raise it to the x power, and you get c:

$$\log_b c = x \quad \Longleftrightarrow \quad b^x = c$$

This rule says that two statements: $x = \log_b c$ and $c = b^x$ are equivalent.

Logarithms were invented by Scottish mathematician John Napier in early 1600, and the notation log was introduced by German mathematician Gottfried Leibniz in 1675.

In some cases, it is easier to operate with exponential expressions than with logarithms. We will see this in the following example.

Example 23.2. Calculate or simplify

a) $\log_2 8$

b) $\log 100$

c) $\log 0.0001$

d) $\log_b 1$

e) $\log_b b$

f) $\log_b b^n$

g) $b^{\log_b c}$

Solution.

a) The problem for computing $\log_2 8$ is, in fact, the same as example 23.1, but written in a different form. Indeed, if $x = \log_2 8$, then, by the circular rule, $2^x = 8$. From example 23.1 we have $x = 3$, so $\log_2 8 = 3$.

b) Let $x = \log 100 = \log_{10} 100$. By the circular rule, $10^x = 100 = 10^2$, so $x = \log 100 = 2$.

c) Let $x = \log 0.0001$. Then $10^x = 0.0001 = 10^{-4}$, so $x = \log 0.0001 = -4$.

d) Let $x = \log_b 1$. Then $b^x = 1 = b^0$. Therefore, $x = \log_b 1 = 0$.

e) Let $x = \log_b b$. Then $b^x = b = b^1$. Therefore, $x = \log_b b = 1$.

f) Let $x = \log_b b^n$. Then $b^x = b^n$. Therefore, $x = \log_b b^n = n$.

g) At the first glance, the expression $b^{\log_b c}$ looks rather complicated. However, if you look closely at this, you will realize that it is actually a very simple. Indeed, if we denote $\log_b c = x$, then $b^x = c$, so, $b^{\log_b c} = b^x = c$.

As you saw in the examples above, they were easily solved after we denoted the logarithm by a single letter (for instance, by *x*), and then converted the logarithmic expression into an exponential one. We will use this method frequently.

Note. Try to memorize answers of problems 23.2, d) and e): for any base *b*,

$$\log_b 1 = 0 \text{ and } \log_b b = 1$$

Example 23.3. Prove that $\log_{\frac{1}{b}} c = -\log_b c$.

Solution. Let's use letters x and y for the above logs: $x = \log_{\frac{1}{b}} c$, $y = \log_b c$. Then

$\left(\dfrac{1}{b}\right)^x = \dfrac{1}{b^x} = b^{-x} = c$, $b^y = c$. From here, $b^{-x} = b^y$. So, $y = -x$, or $x = -y$. Therefore,

$\log_{\frac{1}{b}} c = -\log_b c$.

Basic properties of logarithms

Multiplication Rule: $\log_b (x \cdot y) = \log_b x + \log_b y$.

In words: the logarithm of the product is equal to the sum of the logarithms.

The proof of this statement can be done in a manner similar to examples 23.2 and 23.3. Denote each of the three logs by letters: $A = \log_b(x \cdot y)$, $B = \log_b x$, and $C = \log_b y$. Next, use the circular rule to convert them into exponential form: $b^A = x \cdot y$, $b^B = x$, and $b^C = y$. Now, multiply the second and third equations: $b^B \cdot b^C = x \cdot y$, or $b^{B+C} = x \cdot y$. Compare this equation with $b^A = x \cdot y$. From here, $b^A = b^{B+C}$, hence $A = B + C$, or $\log_b(x \cdot y) = \log_b x + \log_b y$.

Note. Before the era of calculators, there was a widely used device, the so-called logarithmic ruler, or slide ruler, that allowed multiplying numbers based on the Multiplication Rule for logarithms. Schematically, this device works like this. It contains two rulers that allow us to convert numbers into logs and vice versa. To multiply two numbers, the device converts them into logs and adds them up. According to the Multiplication Rule, this sum is the log of the product. Then the device converts this log of the product back to the product of the given numbers. So, the device performs (physically) summation, but mathematically we get multiplication.

Example 23.4. Solve the equation $\log_2(x-1) + \log_2(x-3) = 3$.

Solution. Using the Multiplication Rule, we can combine both logs into one: $\log_2(x-1)(x-3) = 3$. From here, $(x-1)(x-3) = 2^3 = 8$. This is a quadratic equation that can be written in standard form $x^2 - 4x - 5 = 0$. We can solve it by factoring: $(x-5)(x+1) = 0$, and we get two solutions: $x = 5$ and $x = -1$. Let's check these solutions with the original equation. Let $x = 5$. Then

$$\log_2(5-1) + \log_2(5-3) = \log_2 4 + \log_2 2 = 2 + 1 = 3,$$

so everything is OK with this. Now, let $x = -1$. We obtain the logs of negative numbers: $\log_2(-2)$ and $\log_2(-4)$. Such logs do not make sense (they are undefined). Therefore, we must reject the value $x = -1$. Final answer: the equation has the only solution $x = 5$.

Notes.

1) Example 23.4 shows that we need to be very careful when solving log equations: we must check the answer with the original equation. If x and y are both negative, then the left side of the equation $\log_b(x) + \log_b(y) = \log_b(xy)$ is undefined, while the right side is defined. So, when we combine the left side to get the right side, we can get extraneous (incorrect) root.

2) Keep in mind that $\log_a x + \log_b y \neq \log_b(x \cdot y)$ if $a \neq b$. The Multiplication Rule only applies to logs with the same base.

Quotient (or Division) Rule: $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$.

In words: the logarithm of the quotient is equal to the difference of the logarithms.

The proof is similar to that given for multiplication rule. The proof of this statement as well as others statements given below, are left as exercises.

Note. The above two rules: the multiplication and division rules for logs, can be considered as the inverse rules for multiplication and division of exponential expression. If we multiply exponential expressions, we add their powers (powers are logs), if we add logs, we multiply their numbers. If we divide exponential expression, we subtract their powers, if we subtract logs, we divide their numbers. In the next session, we will discuss more about this "inverse connection".

Example 23.5. Solve the equation $\log(x+8)-\log(4x-7)=1$.

Solution. Using the Division Rule, we can represent the left part as the logarithm of the quotient, and the equation takes the form $\log\dfrac{x+8}{4x-7}=1$. From here $\dfrac{x+8}{4x-7}=10^1=10$. Solving this equation, we get $x=2$. Let's check this answer with the original equation:

$$\log(2+8)-\log(4\times 2-7)=\log 10-\log 1=1-0=1. \text{ So, } x=2 \text{ is a solution.}$$

Power Rule: $\log_b x^n = n\cdot\log_b x$.

In words: the logarithm of an exponential expression is equal to its power times the logarithm of its base (the base x of the exponential expression x^n, not the base of log).

Example 23.6. Calculate without using a calculator:

$$3\cdot\log_6 3-\log_6 75+2\cdot\log_6 10$$

Solution. Here we can use all three rules listed above. We can modify the first and third terms like this:

$$3\cdot\log_6 3 = \log_6 3^3 = \log_6 27$$

$$2\cdot\log_6 10 = \log_6 10^2 = \log_6 100$$

From here

$$3\cdot\log_6 3-\log_6 75+2\cdot\log_6 10 = \log_6 27-\log_6 75+\log_6 100$$

$$=\log_6\frac{27\cdot 100}{75}=\log_6 36=\log_6 6^2 = 2\cdot\log_6 6=2.$$

Some scientific calculators allow us to calculate logarithms with only specific bases: base 10 (common logs), and base e (this is a special constant number which we will discuss later in session 25). Logs with the base e are denoted with the symbol **ln** and are called the **natural logarithms**. So

$$\boxed{\ln x = \log_e x}$$

To calculate logs with other bases, we need a way to convert logs from one base to another. The following rule allows us to do this.

Change-of-Base Rule:

$$\log_b x = \frac{\log_d x}{\log_d b} \, .$$

If we need to calculate a logarithm with the base b, but our abilities are limited only by the base d, we can perform a conversion from the base b to d using the Change-of-Base Rule, and then make the calculations. If we put $x = d$, we get a special case of the Change-of-Base Rule:

$$\log_b d = \frac{1}{\log_d b} \, .$$

To prove the Change-of-Base Rule, denote $y = \log_b x$ and convert it into exponential form $b^y = x$. Now apply log with the base d to both sides and use the power rule:

$$\log_d b^y = \log_d x \quad \Longrightarrow \quad y \log_d b = \log_d x \quad \Longrightarrow \quad y = \frac{\log_d x}{\log_d b} \quad \Longrightarrow \quad \log_b x = \frac{\log_d x}{\log_d b}.$$

Example 23.7. Assume that you have a calculator that allows you to calculate common logs only (logs with the base of 10). Calculate $\log_3 5$.

Solution. Using the Change-of-Base Rule and a calculator, we have

$$\log_3 5 = \frac{\log 5}{\log 3} \approx \frac{0.699}{0.477} \approx 1.465 \, .$$

Logarithms are often useful in solving exponential equations in which the powers of exponential expressions are unknown.

Example 23.8. Solve the equation $2^x = 6$ that we discussed at the beginning of this session. Approximate the solution to the nearest thousandth.

Solution. We take the logarithm of both sides of this equation to the base 10 (common log): $\log 2^x = \log 6$. Using the Power Rule, $x \cdot \log 2 = \log 6$, and $x = \frac{\log 6}{\log 2}$. This is the "exact answer". Using a calculator, we can get a numerical approximation:

$$\frac{\log 6}{\log 2} \approx \frac{0.7781}{0.3010} \approx 2.585 \, . \text{ So, } x \approx 2.585.$$

Example 23.9. Solve the equation $3^{2x-1} = 5$. Approximate the solution to the nearest hundredth.

Solution. As in example 23.8, we take the log of both sides: $\log 3^{2x-1} = \log 5$. Using the Power Rule, we have $(2x-1)\log 3 = \log 5$. From here,

$$2x-1=\frac{\log 5}{\log 3}, \quad 2x=\frac{\log 5}{\log 3}+1, \quad x=\frac{1}{2}\left(\frac{\log 5}{\log 3}+1\right)\approx\frac{1}{2}\left(\frac{0.699}{0.477}+1\right)\approx 1.23.$$

So, $x \approx 1.23$.

Note. The method of taking the logarithm of both sides of a given equation, which we used in examples 23.8 and 23.9, is very often used for equations containing exponents. Theoretically, we can take the logarithm with any base. We used common logs (logs with the base 10) to be able to use a calculator. We could also use natural logs (logs with the base e) since most calculators allow us to calculate logs with this base.

Exercises 23

In exercises 23.1 and 23.2, convert the given statements from logarithmic form to exponential form.

23.1. a) $\log_2 16 = 4$ 23.2. a) $\log_4 64 = 3$

b) $\log_2 \left(\dfrac{1}{16} \right) = -4$ b) $\log_4 \left(\dfrac{1}{64} \right) = -3$

In exercises 23.3 and 23.4, convert the given statements from exponential form to logarithmic form (use the bases of exponential expressions as the bases of logs).

23.3. a) $2^3 = 8$ 23.4. a) $3^4 = 81$

b) $2^{-3} = \dfrac{1}{8}$ b) $3^{-4} = \dfrac{1}{81}$

In exercises 23.5 and 23.6, calculate **without** using a calculator.

23.5. a) $\log_3 81$ 23.6. a) $\log_4 16$

b) $\log 1000$ b) $\log 10{,}000$

c) $\log 0.01$ c) $\log 0.001$

d) $\log_{\frac{1}{2}} 64$ d) $\log_{\frac{1}{3}} 27$

e) $\log_6 \dfrac{1}{36}$ e) $\log_4 \dfrac{1}{16}$

f) $\log_4 \sqrt[5]{4}$ f) $\log_6 \sqrt[4]{6}$

g) $\log_3 \left(9\sqrt[4]{3} \right)$ g) $\log_7 \left(49\sqrt[5]{7} \right)$

In exercises 23.7 – 23.10, solve the given equations.

23.7. a) $3^x = 27$ 23.8. a) $5^x = 625$

b) $4^{3x-1} = \dfrac{1}{64}$ b) $6^{2x-3} = \dfrac{1}{36}$

c) $\left(\dfrac{1}{5} \right)^{2x+3} = 125$ c) $\left(\dfrac{1}{3} \right)^{3x+2} = 81$

23.9. a) $\log_3(x-4)+\log_3(x+2)=3$ 23.10. a) $\log_4(x+4)+\log_4(x-2)=2$

    b) $\log_5(7x-3)-\log_5(x-3)=2$ b) $\log_2(15x+3)-\log_2(x+3)=3$

In exercises 23.11 and 23.12, calculate **without** using a calculator.

23.11. $\log_8 32 - 2\cdot\log_8 6 + \log_8 9$ 23.12. $3\cdot\log 20 + 2\cdot\log 3 - \log 72$

In exercises 23.13 and 23.14, let $u = \ln x$ and $v = \ln y$. Write given expressions in terms of u and v.

23.13. $\ln\left(x^3 y^2\right)$ 23.14. $\ln\left(x^4 y^5\right)$

In exercises 23.15 and 23.16, let $u = \log x$ and $v = \log y$. Write given expressions in terms of u and v.

23.15. $\log\left(\dfrac{x^7}{y^6}\right)$ 23.16. $\log\left(\dfrac{y^3}{x^2}\right)$

In exercises 23.17 and 23.18, write the explression as a single logarithm.

23.17. a) $3\log(x) - 2\log(y) + \frac{2}{3}\log(z)$ 23.18. a) $4\log(x) - 3\log(y) + \frac{4}{5}\log(z)$

    b) $5\log(x) + 6\log(y) - \frac{3}{4}\log(z)$ b) $6\log(x) + 7\log(y) - \frac{5}{6}\log(z)$

In exercises 23.19 and 23.20, solve the given equations. Round your answers to the nearest thousands.

23.19. a) $6^x = 9$ 23.20. a) $5^x = 8$

    b) $7^{3x+2} = 4$ b) $4^{5x-3} = 7$

    c) $e^{4x+3} = 5$ c) $e^{3x-4} = 6$

In exercises 23.21 and 23.22, assume that a calculator allows us to calculate only common logs (logs with the base 10) and natural logs (logs with the base e). Calculate the given expressions using both types of logs. Round your answers to the nearest thousands. Compare the results when using "log" and "ln".

23.21. $\log_7 8$ 23.22. $\log_8 7$

Challenge Problems

23.23. Prove that $\log_b\left(\dfrac{1}{a}\right) = \log_{\frac{1}{b}}(a)$.

23.24. Prove that $a^{\log b} = b^{\log a}$.

23.25. Calculate without calculator

$$\frac{1}{\log_2(6)} + \frac{1}{\log_3(6)}$$

23.26. Consider the equation

$$\log_c(x+a) + \log_c(x+b) = p+q, \quad \text{where } b = a - c^p + c^q.$$

Prove that the only solution of this equation is $x = c^p - a$.

Session 24

Exponential and Logarithmic Functions

We already studied some functions: quadratic functions (parabolas) and trigonometric functions. In this session, we will study exponential expressions and logarithms from the point of view of functions. For these functions, we will denote by the letter **a** any positive number, not equal to 1, and call it the **base** of a function.

Exponential Functions

We can treat the expression a^x as a function of x: if we pick any number x, the expression will produce the value $y = a^x$. We can also write $f(x) = a^x$. The domain of this function (the set of possible values of x) is the set of all real numbers (since any number x can be taken as a power, so there are no exceptions), but the range (the set of possible values of y) is the set of only positive numbers (since the value of a^x cannot be negative number or zero).

We are interested in the **behavior** of this function. It means that we want to know what happens with the value y when x takes some specific values, when x increases to positive infinity, or decreases to negative infinity. One of the ways to study a function is to visualize it, in other words, construct its **graph**. One point on the graph is easy to observe: if $x = 0$, then $y = a^0 = 1$. So, for any base a, the graph of the exponential function $y = a^x$ passes through the point (0, 1) which is located on the y-axis one unit above the origin. It turns out that the shape of the graph of the function $y = a^x$ depends on whether the base a is greater or less than 1. (Case $a = 1$ is not interesting since $1^x = 1$ for any x, and the graph of $y = 1$ is a horizontal line).

1. Case $a > 1$. In this case, the larger x, the larger y. We say that the function $y = a^x$ **increases**, and it increases very fast. For example, if we take $a = 2$, we can construct the following tables of values of the function $y = 2^x$ for non-negative and negative values of x:

x	0	1	2	3	4	5
$y = 2^x$	1	2	4	8	16	32

Non-negative value of x

x	-1	-2	-3	-4	-5
$y = 2^x$	$2^{-1} = \dfrac{1}{2}$	$2^{-2} = \dfrac{1}{4}$	$2^{-3} = \dfrac{1}{8}$	$2^{-4} = \dfrac{1}{16}$	$2^{-5} = \dfrac{1}{32}$

Negative values of x

Based on these two tables we can draw the graph of the function $y = 2^x$:
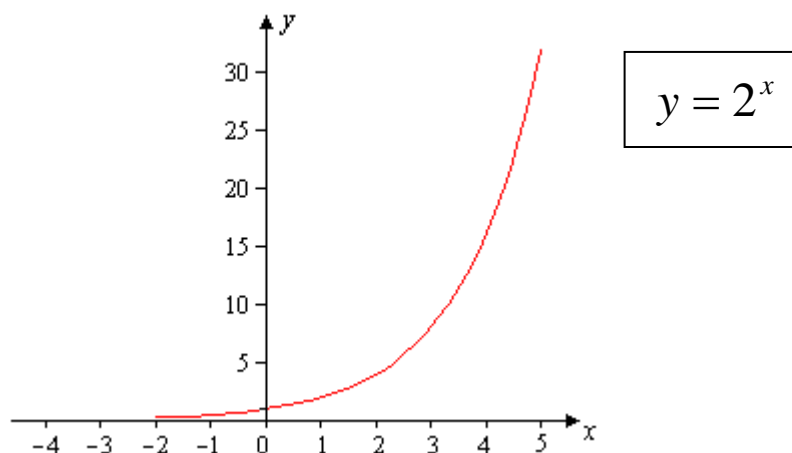
Notice that when x goes to positive infinity (moving to the right), y also goes to positive infinity (moving up), and when x goes to negative infinity (moving to the left), y approaches to zero (approaches to the x-axis but never touches it). We say that the x-axis is the horizontal **asymptote** of the function $y = a^x$.

2. Case $0 < a < 1$. In this case, the larger x, the smaller y. We say that the function $y = a^x$ **decreases**. Let's take as an example $a = \dfrac{1}{2}$. We can graph the function $y = \left(\dfrac{1}{2}\right)^x$ in the same way as we did above for the function $y = 2^x$ by creating a table of its values. However, we can get this graph almost immediately, if we notice that $\left(\dfrac{1}{2}\right)^x = \dfrac{1}{2^x} = 2^{-x}$. So, we actually need to graph the function $y = 2^{-x}$.

Let's consider in general the relationship between the graphs of the functions $f(x)$ and $f(-x)$. Points (x, y) and $(-x, y)$ are symmetrical to each other with respect to the y-axis. Therefore, the graphs of $f(x)$ and $f(-x)$ are also symmetrical to each other with respect to the y-axis. This means that if we have already drawn the graph of $f(x)$, then to get the graph of $f(-x)$, we can just reflect the graph of $f(x)$ with respect to the y-axis.

We can apply the above reasoning to the function $y = \left(\dfrac{1}{2}\right)^x = 2^{-x}$ and reflect the graph of $y = 2^x$ with respect to the y-axis. Here is the resulting picture
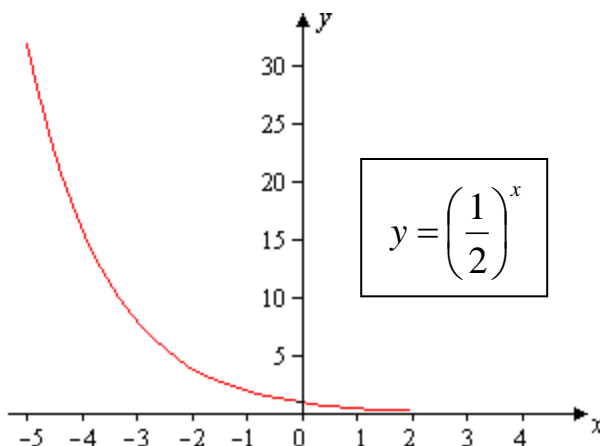
As you can see, when x goes to negative infinity, y increases to positive infinity, and when x goes to positive infinity, y approaches to zero (but doesn't equal to it), so the x-axis is still horizontal asymptote of the function $y = \left(\dfrac{1}{2}\right)^x$.

Logarithmic Functions

Similar to exponential expressions, we can treat the logarithm $\log_a x$ (with fixed base a) as a function of x: $y = \log_a x$. Its domain (the set of possible values of x) is the set of all positive numbers, and range (the set of possible values of y) is the set of all real numbers.

Note. Notice that the domain of $\log_a x$ is the same as the range of a^x, and the range of $\log_a x$ is the same as the domain of a^x. This is not a coincidence: we will see shortly that this is related to the concept of inverse functions.

Example 24.1. Find the domain of the following functions:

$$\text{a) } y = \log_a(3x+1) \qquad\qquad \text{b) } y = \log_a(5-2x)$$

Solution. Since the domain of the function $y = \log_a x$ is the set of all positive numbers, to find domains of given functions, we just need to solve the inequalities by making expressions in parentheses greater than zero:

a) $3x+1>0 \implies 3x>-1 \implies x>-1/3$. So, the domain is the interval $(-1/3, \infty)$.

b) $5-2x>0 \implies -2x>-5 \implies x<5/2$. So, the domail is the interval $(-\infty, 5/2)$.

All logarithmic functions (for any base a) have the same value of zero at $x = 1$:

$$\log_a 1 = 0.$$

Thus, the graphs of all logs pass through the same point $(1, 0)$, located on the x-axis one unit to the right of the origin. Similar to exponential functions, the shape of the graph of the function $y = \log_a x$ depends on whether a is greater or less than 1.

1. Case $a > 1$. In this case, as for the exponential function, the log function increases, but this time it increases very slowly. Let's take as an example $a = 2$. Here are the tables of values of the function $y = \log_2 x$ for values of x greater and less than 1.

x	1	2	4	8	16	32
$y = \log_2 x$	0	1	2	3	4	5

Values of x greater than 1

x	$\dfrac{1}{2}$	$\dfrac{1}{4}$	$\dfrac{1}{8}$	$\dfrac{1}{16}$	$\dfrac{1}{32}$
$y = \log_2 x$	-1	-2	-3	-4	-5

Values of x less than 1

The graph of the function $y = \log_2 x$ is this

$$y = \log_2 x$$
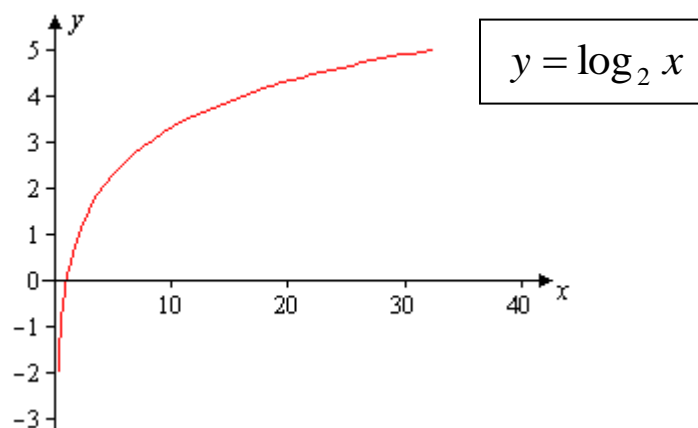
Here the y-axis is the vertical asymptote: y tends to negative infinity when x approaches to zero, but never touches the y-axis.

2. Case $0 < a < 1$. In this case, like for the exponential function, the function $y = \log_a x$ decreases, but now it decreases very slowly. Let's take the example of $a = \dfrac{1}{2}$. To draw a graph of $y = \log_{\frac{1}{2}} x$ without creating a table of values, we can use the result of example 23.3 from the previous session: $\log_{\frac{1}{a}} x = -\log_a x$. In our particular case,

$$\log_{\frac{1}{2}} x = -\log_2 x.$$

Let's consider in general the relationship between the graphs of the functions $y = f(x)$ and $y = -f(x)$. The points (x, y) and $(x, -y)$ are symmetrical to each other with respect to

the x-axis. Therefore, the graphs of $f(x)$ and $-f(x)$ are also symmetrical to each other with respect to the x-axis. So, to get the graph of $-f(x)$, we can simply reflect the graph of $f(x)$ about the x-axis. You may recall that we discussed such relationship in session 11 when we compared the graphs of the parabolas $y = x^2$ and $y = -x^2$.

Applying this reasoning to the function $y = \log_{\frac{1}{2}} x = -\log_2 x$, we get the picture

$$y = \log_{\frac{1}{2}} x$$
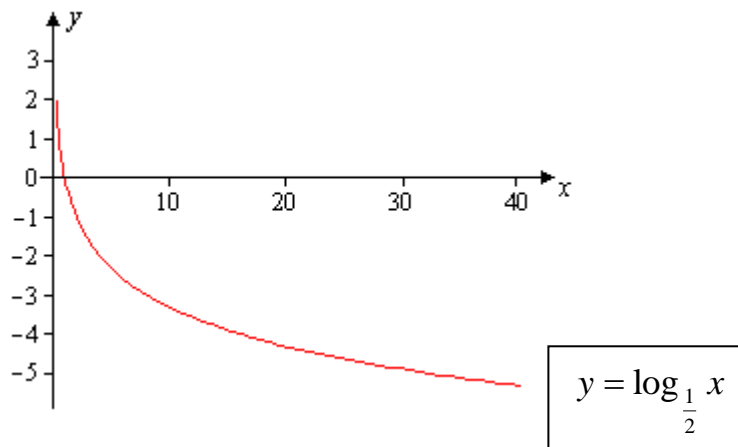
Here the y-axis remains the vertical asymptote.

Relation between Exponential and Logarithmic Functions

If you compare the tables of values of functions $y = 2^x$ and $y = \log_2 x$, shown above, you may notice that the variables x and y exchange their values. This is not a coincidence. The fact is that the functions $y = a^x$ and $y = \log_a x$ are **inverse** to each other. Let's consider this concept in general form.

Let $y = f(x)$ be a function. As we already mentioned, we can treat variable x as input that goes into function f, then f operates on x and produces the output y. Schematically, we can represent the function f by the diagram
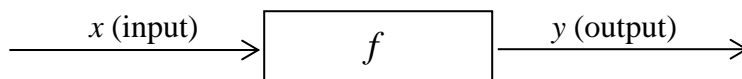
A function g is called the **inverse** to f, if it does the job opposite to f: it passes y back to x. In other words, the input of the inverse function is y, and the output is x. We usually denote the function inverse to f by f^{-1}. Schematically, we can represent the inverse function f^{-1} by the diagram
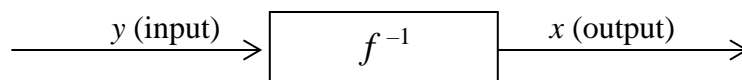
Note. The notation f^{-1} for inverse function may create confusion with the notation $\dfrac{1}{f}$ for

the reciprocal function. Keep in mind that these are completely different functions.

To find the function inverse to f, we can solve the equation $y = f(x)$ for x, and then exchange x and y: replace x with y, and y with x.

Example 24.2. Find the inverse function to $y = x^2$, where $x \geq 0$.

Solution. If we solve the equation $y = x^2$ for non-negative x, we get $x = \sqrt{y}$. Now, just exchange x and y. The inverse function is $y = \sqrt{x}$.

Example 24.3. Find the inverse function to $y = \log_a x$.

Solution. As in example 24.2, we solve the equation $y = \log_a x$ for x. Using the "Circular Rule" described in session 23, we have $x = a^y$. Now, exchange x and y, and get the inverse function $y = a^x$.

As you can see, the logarithmic and exponential functions are **inverse** to each other.

Let's return to a general case of the function f, and see how the graphs of f and f^{-1} are related. If $(x, f(x))$ is a point on the graph of f, then the point $((f(x), x)$ will be on the graph of f^{-1}. The points with coordinates (a, b) and (b, a) are symmetrical to each other with respect to the line $y = x$, which is the bisector of the first and third quadrants. (To see that fact, you can look at some examples, like points (3, 4) and (4, 3), or try to prove this statement in a general form). Therefore, the graphs of the function f and its inverse f^{-1} are symmetrical to each other with respect to the line $y = x$.

Let's draw together the graphs of the functions $y = 2^x$ and $y = \log_2 x$ which are inverse to each other:
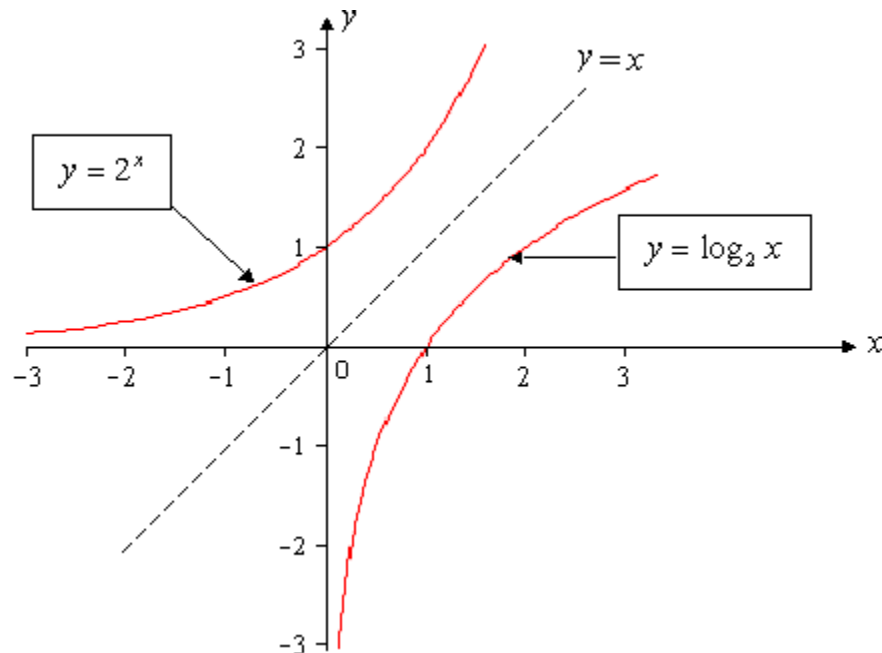
Exercises 24

In exercises 24.1 – 24.8, graphs of the given functions f and g are drawn. One of the graphs is labeled as A and another as B. Match the functions f and g with the graphs. Explain your answers.

24.1. $f(x) = 3^x$, $g(x) = 4^x$.
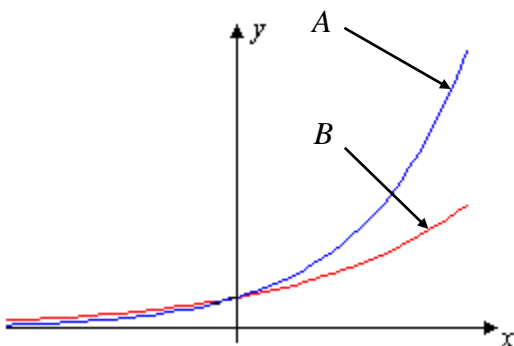
24.2. $f(x) = 4^x$, $g(x) = 5^x$.
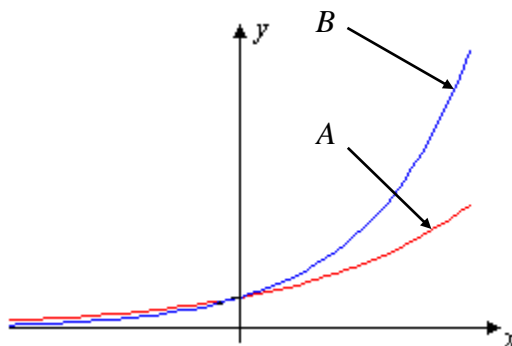
24.3. $f(x) = \left(\dfrac{1}{3}\right)^x$, $g(x) = \left(\dfrac{1}{4}\right)^x$.
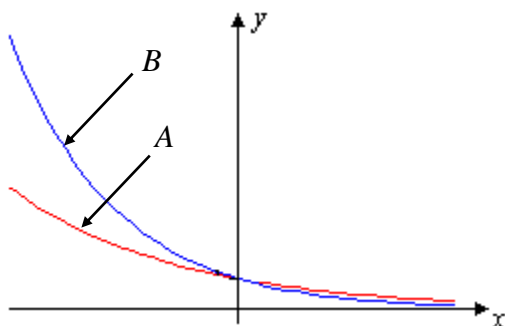
24.4. $f(x) = \left(\dfrac{1}{4}\right)^x$, $g(x) = \left(\dfrac{1}{5}\right)^x$.
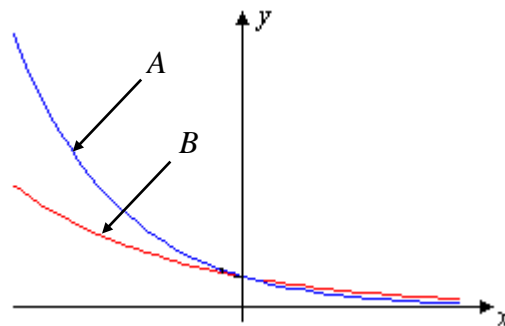
24.5. $f(x) = \log_3(x)$, $g(x) = \log_4(x)$.
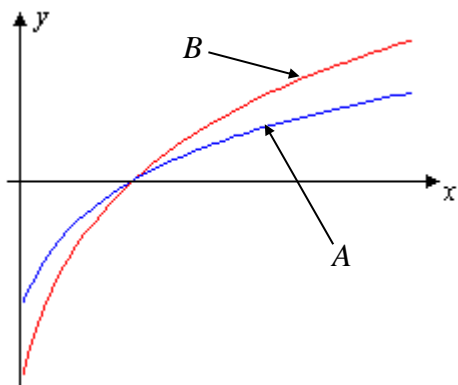
24.6. $f(x) = \log_4(x)$, $g(x) = \log_5(x)$.
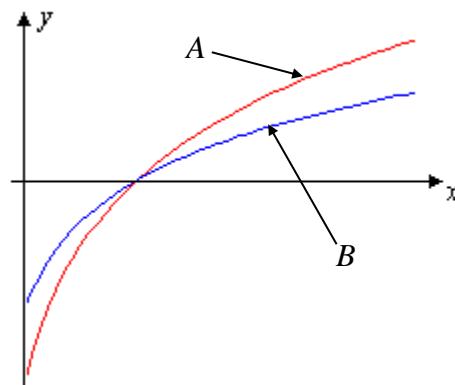
24.7. $f(x) = \log_{\frac{1}{3}}(x), \ g(x) = \log_{\frac{1}{4}}(x)$.
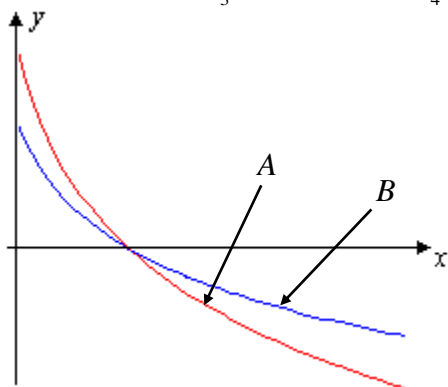
24.8. $f(x) = \log_{\frac{1}{4}}(x), \ g(x) = \log_{\frac{1}{5}}(x)$.
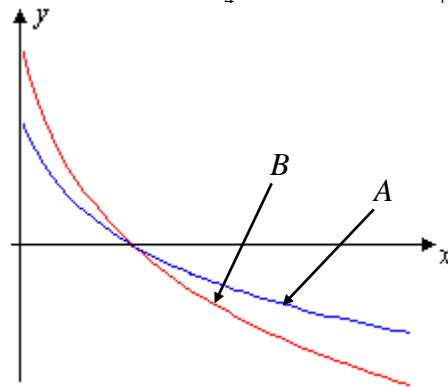
In exercises 24.9 and 24.10, find the inverse function to the function $f(x)$.

24.9. a) $f(x) = 3^x$

   b) $f(x) = \log_{\frac{2}{3}}(x)$

24.10. a) $f(x) = \left(\dfrac{3}{4}\right)^x$

   b) $f(x) = \log_5(x)$

In exercises 24.11 and 24.12, find the domains of the given functions.

24.11. a) $y = \log_a(2x+3)$

   b) $y = \log_a(6-4x)$

24.12. a) $y = \log_a(4x+5)$

   b) $y = \log_a(10x-8)$

Session 25

Compound Interest and Number *e*

If you deposit money to a bank, the bank pays you interest for using your money. If you borrow money from a bank, you pay interest to the bank. The interest may be **simple** or **compound**. To explain these, let's start with some terminology and notation.

P – Principal or Initial Value. This is the amount of money you deposit to the bank or borrow from the bank.

T – Time. This is the period of time during which the money is used (by you or the bank). In calculations, it is usually calculated in years. If the time period is several months, it is given by fraction or decimal. For example, $T = 0.5$ means 6 months.

R – Rate. This is the interest rate used to pay for the use of money. This is usually indicated as a percentage per year. In calculations, it is used as a decimal. For example, if interest rate is 1.7%, in calculations it is used as 0.017.

I – Interest. This is the amount of money that you (or the bank) earn for using money for *T* years. (Don't confuse interest with rate: interest is dollar amount, while rate is percentage).

A – Amount or Future Value. This is the amount of money that you will have in *T* years. The future value is the sum of two parts: Initial Value and Interest. So, $A = P + I$.

Simple Interest

This type of interest is usually used when you keep money for a short period of time, such as a few months. This interest is really simple to calculate. If you invest *P* dollars for one year at the rate of *R* (taking it as a decimal), then the interest *I* (that's what you earn), is $I = PR$. If you keep money for *T* years, the total interest earned by you will be $I = PRT$. This is a formula of simple interest. As you can see, it's really simple. We can also calculate the future value *A*: $A = P + I = P + PRT = P(1 + RT)$. So, the basic formulas related to simple interest are

$$I = PTR, \quad A = P(1+RT)$$

Note. When using the above formulas, keep in mind that the rate *R* must be taken as a decimal (not as a percentage), and the time *T* should be in the same units of time as the rate *R* (usually in years).

Example 25.1. Suppose you deposit $800 for 3 months into a bank that pays 5% of simple interest. Calculate the interest that the bank will pay you and the future value (the amount you will withdraw) after 3 months.

Solution. We have

$$P = 800, \quad R = 5\% = 0.05, \quad T = 3\,\text{months} = \frac{3}{12}\,\text{years} = 0.25.$$

Using the above formulas for I and A, we get

$I = PRT = 800 \times 0.05 \times 0.25 = 10$ (dollars), $\quad A = P + I = 800 + 10 = 810$ (dollars).

Compound Interest

If you keep money in a bank for a long period of time (for example on CD – Certificate of Deposit, for several years), it is unfair to calculate interest using the above formula for simple interest. Indeed, if the principal is P, and the bank rate is R, then the amount A_1 after the first year, according to the formula for future value with $T = 1$, is $A_1 = P(1 + R)$. Assume that after the first year you do not withdraw your money. Then for the second year it would be unfair to take as a principal the original value P. Instead, it is reasonable to take the value of A_1 (which is, of course, greater than P) as a new principal. In other words, for the second year, the rate R should be applied not only to the initial deposit P, but also to the interest $I = PR$ that you earned for the first year. Notice that according to the formula $A_1 = P(1 + R)$, the amount at the end of a year is equal to the amount at the beginning of the year times $(1 + R)$. Therefore, at the end of the second year the amount, denoted as A_2, should be expressed thorough A_1 by the same rule:

$$A_2 = A_1(1 + R) = P(1 + R)(1 + R) = P(1 + R)^2.$$

If we continue the same reasoning, then in T years you will accumulate the amount of $A_T = P(1 + R)^T$ dollars. Formula

$$A = P(1 + R)^T$$

allows us to calculate the future value after T years, if the initial principal is P, and the bank interest is R.

Although the above formula for future value is fairer than the corresponding formula for simple interest, it is still not fair enough. Here is the reasoning. After a certain (even a short) period of time, say, in half a year, your principal will be increased by earned interest during this period. However, according to the above formula, the original principal remains unchanged throughout the year, and only at the beginning of the next year the bank recalculates and replaces the original principal with a new value. It would be better (for customers) if such recalculations would be made more often. Many banks do that. They introduce a parameter called the **compounding period**. This is the period of time after which the bank recalculates the principal: the bank takes the principal, adds the earned interest and uses this sum as a new principal. Usually, bank compounds (recalculates) semiannually (every half of a year), quarterly (every three months), monthly, and even daily. Therefore, the above formula for the future value should be modified by including a new parameter N – the number of compounding periods per year.

If interest is compounded yearly, then $N = 1$; if semi-annually, then $N = 2$; quarterly, then $N = 4$; monthly, then $N = 12$; weekly, then $N = 52$; daily, then $N = 365$.

Let's modify the above formula for future value if the investment is compounded monthly, i.e. $N = 12$. Since the rate R is constant throughout the year, interest for one month will be

$I = \dfrac{PR}{12}$. After the 1$^{\text{st}}$ month, future value is $A = P + I = P + \dfrac{PR}{12} = P\left(1 + \dfrac{R}{12}\right)$. So, in order to get the future value for any month, we should take the future value for the previous month and multiply it by the expression $1 + \dfrac{R}{12}$. Therefore, after the 2$^{\text{nd}}$ month, the future value is

$$A = P\left(1 + \frac{R}{12}\right)\left(1 + \frac{R}{12}\right) = P\left(1 + \frac{R}{12}\right)^2 .$$

At the end of the 1$^{\text{st}}$ year, the future value becomes $A = P\left(1 + \dfrac{R}{12}\right)^{12}$.

And, at the end of the T years, the future value becomes $A = P\left(1 + \dfrac{R}{12}\right)^{12T}$. In similar way, for any the number of compounding periods per year N, we can get the **general compound interest formula**:

$$A = P\left(1 + \frac{R}{N}\right)^{TN}$$

Here $\dfrac{R}{N}$ is the rate for one compounding period, and TN is the total number of compounding periods for T years. For example, if rate $R = 2.4\%$, deposit is compounded quarterly (so, $N = 4$), and the number of years $T = 5$, then $\dfrac{R}{N} = \dfrac{0.024}{4} = 0.006$ and $TN = 5 \cdot 4 = 20$.

Interest I on this deposit is the difference between future value A and the original principal P:

$$I = A - P = P\left(1 + \frac{R}{N}\right)^{TN} - P = P\left[\left(1 + \frac{R}{N}\right)^{TN} - 1\right] .$$

Example 25.2. Suppose you deposit \$300 for 8 years at 3% compounded quarterly. Find the future value and earned interest.

Solution. We have: $P = 300$, $R = 3\% = 0.03$, $N = 4$, $T = 8$. Substitute these values into the compound interest formula and calculate future value A:

$$A = 300 \cdot \left(1 + \frac{0.03}{4}\right)^{8 \cdot 4} = 300 \cdot (1.0075)^{32} \approx 300 \cdot 1.2701 = 381.03 .$$

So, the future value is \$381.03. Interest I is the difference:

$$I = A - P = 381.03 - 300 = 81.03.$$

Therefore, you will earn \$81.03 for 8 years.

The compound interest formula can be used to find the rate R, or time T needed to accumulate the desired amount in the future.

Let's solve the problem to find time T in general form. Dividing both parts of the compound interest formula by P, we have

$$\frac{A}{P} = \left(1 + \frac{R}{N}\right)^{TN}.$$

Now, take log from both sides: $\log \frac{A}{P} = \log\left(1 + \frac{R}{N}\right)^{TN} = TN \cdot \log\left(1 + \frac{R}{N}\right).$

From here,

$$T = \frac{\log \dfrac{A}{P}}{N \cdot \log\left(1 + \dfrac{R}{N}\right)}.$$

Note. You do not need to memorize this formula. To find time in particular problems, plug in given data into the compound interest formula, and apply log to both sides.

Example 25.3. Suppose you deposit some amount of money at 6% compounded monthly. In how many years your deposit will be doubled?

Solution. According to the problem, the future value A is twice as the principal P: $A = 2P$. Also, $R = 6\% = 0.06$, $N = 12$. Substitute these data into the compound interest formula:

$$A = P\left(1 + \frac{R}{N}\right)^{TN} \Rightarrow 2P = P\left(1 + \frac{0.06}{12}\right)^{12T}.$$

Reduce (divide) both side by P and calculate the expression inside parentheses: $2 = (1.005)^{12T}$. Now apply log to both sides

$$\log 2 = \log(1.005)^{12T} \Rightarrow \log 2 = 12 \cdot T \cdot \log(1.005).$$

From here $T = \dfrac{\log 2}{12 \cdot \log(1.005)} = 11.6$.

.
So, your deposit will be doubled in about 11.6 years.

When you decide which bank to deposit your money to or which credit card to use for making only minimum payments, you need to consider not only the rate, but also the compounding period. To make a true comparison of different rates, we can compare the interest that accrues on one dollar in one year. This value is called the **effective rate** or APY (Annual Percentage Yield). To get the formula for APY, we substitute the values $P = 1$ and $T = 1$ into the formula for the interest I (this formula is above example 25.2). We will have

$$APY = \left(1 + \frac{R}{N}\right)^N - 1.$$

Usually, APY is presented as percentage.

Example 25.4. Suppose you have the option of using two credit cards for which you want to make only minimum payments. On the 1st card, you will pay 18% interest compounded monthly, and on the 2nd card $-$ 17. 9% compounded daily. Which deal is best for you?

Solution. On the first glance, it looks like the 2nd card is better (you pay smaller rate). However, let's compare APYs for these two cards.

1st card: $APY = \left(1 + \dfrac{0.18}{12}\right)^{12} - 1 \approx 0.1956$, or APY $= 19.56\%$.

2nd card: $APY = \left(1 + \dfrac{0.179}{360}\right)^{360} - 1 \approx 0.1960$, or APY $= 19.60\%$.

As you can see, although the rate on the 1st credit card is higher, you would prefer this card because its APY is lower and in the long run you will pay less interest.

Continuous compound interest. Number *e*

We saw that when you invest in a bank, it's more profitable for you, if bank uses the compound interest formula instead of the simple one. Also, the shorter the compounding period, the greater your profit. We mentioned the cases of compounding semiannually, quarterly, monthly, and daily. But why should we be limited only with to these periods? Can the compounding period to be one hour, one minute, or even one second? The answer is yes. In this way we come up to a formula called **continuous compound interest**.

To get this formula, let's modify the compound interest formula:

$$A = P\left(1 + \frac{R}{N}\right)^{TN} = P\left[\left(1 + \frac{R}{N}\right)^{\frac{N}{R}}\right]^{TR}.$$

If we denote $e_N = \left(1 + \dfrac{R}{N}\right)^{\frac{N}{R}}$, then $A = P(e_N)^{TR}$.

Let's see what happens to your income when the compounding period becomes shorter and shorter. In this case, interest is recalculated more often, and, as a result, the future value (i.e. your money) becomes larger. You might think that the future value will grow endlessly and, eventually, can become huge. However (unfortunately for you) this is not so. Your income has a limit. If N increases to infinity (becomes bigger and bigger), then the value $\dfrac{N}{R}$ also increases to infinity. However, this is not the case for e_N (and also for future value

$A = P(e_N)^{TR}$). Although this value increases, it does not increase to infinity. Let's calculate e_N for some values of $\dfrac{N}{R}$.

$\dfrac{N}{R}$	1	2	3	4	5	10	100	1000
e_N	2	2.25	2.37	2.44	2.49	2.59	2.70	2.717

It can be shown (it is done in Calculus courses) that if N continues to increase, the value of e_N cannot be greater than 3. In fact, e_N is getting closer and closer to a certain constant number. This number is denoted by the letter e and is approximately equal to 2.718. This number is called the Euler's number or the **base of the natural logarithms**. Most scientific calculators have a button to calculate the number e with even greater accuracy.

Let's return to the formula $e_N = \left(1 + \dfrac{R}{N}\right)^{\frac{N}{R}}$ and denote power $\dfrac{N}{R} = n$. Then, by taking the reciprocal, $\dfrac{R}{N} = \dfrac{1}{n}$ and $e_N = \left(1 + \dfrac{1}{n}\right)^{n}$. If N is big, number n is also big and $e_N \approx e$. We may say that

$$e \approx \left(1 + \dfrac{1}{n}\right)^{n} \approx 2.718, \text{ if number } n \text{ is big.}$$

If we replace e_N in the formula $A = P(e_N)^{TR}$ with the number e, we will get the **continuous compound interest formula**

$$\boxed{A = Pe^{TR}}$$

This formula gives the maximum possible future value compared to the compound interest formula with any finite number N of compounding periods per year.

Example 25.5. Suppose you invested \$300 for 8 years at 3% compounded continuously. Find the future value and earned interest.

Solution. We have: $P = 300$, $R = 3\% = 0.03$, $T = 8$. Substitute these values into the above formula and calculate future value A:

$$A = Pe^{TR} = 300 \cdot e^{8 \cdot 0.03} = 300 \cdot e^{0.24} = 300 \cdot 1.27125 = 381.37$$

So, the future value is \$381.37. Interest I is the difference:

$$I = A - P = 381.37 - 300 = 81.37.$$

Compare this result with the result of example 25.2. The difference in interest is small (34 cents), but for large investments the difference becomes significant.

Exercises 25

Round all answers to the nearest cent.

25.1. Suppose you deposit $700 for 5 months into a bank that pays 2.5% simple interest. Calculate interest that bank will pay you and future value (amount that you can withdraw) after 5 months.

25.2. Suppose you deposit $600 for 7 months into a bank that pays 4% simple interest. Calculate interest that bank will pay you and future value (amount that you withdraw) after 7 months.

25.3. Suppose you deposit $1,200 for 5 years at 5% compounded monthly. Find the future value and earned interest.

25.4. Suppose you deposit $2,400 for 6 years at 2% compounded semiannually. Find the future value and earned interest.

25.5. Suppose you made an investment of $1,200 for 5 years at 5% compounded continuously. Find the future value and earned interest. Compare with the results of exercise 25.3.

25.6. Suppose you made an investment of $2,400 for 6 years at 2% compounded continuously. Find the future value and earned interest. Compare with the results of exercise 25.4.

25.7. Suppose you deposit a certain amount of money at 4% compounded quarterly. In how many years will your deposit be doubled?

25.8. Suppose you deposit a certain amount of money at 7% compounded daily. In how many years will your deposit be doubled?

Challenge Problem

25.9. Suppose you want to invest $5,000 for the next 10 years. You are considering two banks for your investment. Bank *A* offers a rate of 7 % compounded semiannually. Bank *B* offers a rate of 6.9 % compounded daily. Which bank would you choose? Is it possible to solve this problem without information on the investment amount and time?

Answers to Exercises

Session 1

1.1. a) System has unique solution $(-5,\ 2,\ -1)$

    b) System has unique solution $(-2,\ 1,\ -3)$.

    c) System is dependent. Possible parametric form: $x=t,\ \ y=\dfrac{1}{2}-2t,\ \ z=t$.

       Particular solutions for $t=0$: $\left(0,\ \dfrac{1}{2},\ 0\right)$.

    d) System has unique solution $(0,\ -2,\ 1)$

    e) System is inconsistent: no solutions.

1.2. a) System has unique solution $(4,\ -2,\ 3)$

    b) System has unique solution $(3,\ -4,\ -2)$.

    c) System is inconsistent: no solutions.

    d) System has unique solution $(1,\ 0,\ 2)$

    e) System is dependent. Possible parametric form: $x=1-t,\ \ y=t,\ \ z=1+t$.

       Particular solutions for $t=0$: $(1,\ 0,\ 1)$.

Session 1A

1A.1. $\left(\dfrac{64}{17},\dfrac{52}{17}\right)$ **1A.2.** $\left(\dfrac{55}{23},-\dfrac{17}{23}\right)$ **1A.3.** $D=22$ **1A.4.** $D=-6$

1A.5. $\left(\dfrac{35}{22},\dfrac{37}{22},-\dfrac{9}{22}\right)$ **1A.6.** $\left(\dfrac{11}{6},\dfrac{17}{3},\dfrac{29}{6}\right)$

Session 2

2.1. a) $3x^2+x-14=0;$ $a=3,\ \ b=1,\ \ \ c=-14$

    b) $16x^2-24x+3=0;\ \ a=16,\ b=-24,\ c=3$

2.2. a) $8x^2-2x-3=0;$ $a=8,\ \ \ b=-2,\ \ c=-3$

    b) $25x^2+20x+1=0;\ \ a=25,\ b=20,\ \ \ c=1$

2.3. a) $\left\{-\dfrac{7}{3},2\right\}$ **b)** $\left\{\dfrac{5}{6}\right\}$ **2.4. a)** $\left\{\dfrac{3}{4},-\dfrac{1}{2}\right\}$ **b)** $\left\{-\dfrac{4}{7}\right\}$

2.5. a) $\left\{0,-\dfrac{5}{3}\right\}$ **b)** $\{\pm 4\}$ **c)** $\{-4,3\}$ **d)** $\{-5\}$ **e)** $\{4,-5\}$

Session 2 (*continued*)

2.6. **a)** $\left\{0, \dfrac{7}{6}\right\}$ **b)** $\{\pm 3\}$ **c)** $\{-3, 5\}$ **d)** $\{-6\}$ **e)** $\{6, -4\}$

2.7. **a)** $\left\{\dfrac{3}{4}, -\dfrac{1}{2}\right\}$ **b)** $\left\{\dfrac{3}{5}, -3\right\}$ **c)** $\left\{\dfrac{1}{3}, \dfrac{2}{3}\right\}$

2.8. a) $\left\{\dfrac{1}{3}, -\dfrac{5}{2}\right\}$ **b)** $\left\{\dfrac{4}{7}, -1\right\}$ **c)** $\left\{-\dfrac{1}{4}, \dfrac{2}{3}\right\}$

Session 3

3.1. 2.5×10^4 **3.2.** 2.9×10^4 **3.3.** 2.56×10^{-4} **3.4.** 1.4×10^{-5}

3.5. 0.00004 **3.6.** 0.0875 **3.7.** $3,475,000$ **3.8.** $1,236$

3.9. a) 3.47×10^{-3}, **b)** 2.5×10^2 **3.10. a)** 4.38×10^{-5}, **b)** 3.6×10^4

3.11. a) 2, **b)** 1, **c)** 1/16, **d)** $-$ 1/16, **e)** 1/16

3.12. a) 3, **b)** 1, **c)** 1/25, **d)** $-$ 1/25, **e)** 1/25

3.13. a) $1/a^2$, **b)** $1/c^9$, **c)** $1/a^6$, **d)** n^2/m^2

3.14. a) $1/c^3$, **b)** $1/n^6$, **c)** $1/d^{12}$, **d)** $-b^3/a^3$

3.15. a) p^4, **b)** $1/p^{12}$, **c)** p^{12}, **d)** $1/p^4$, **e)** $1/p^4$, **f)** p^{12}, **g)** $1/p^{12}$, **h)** p^4

3.16. a) r^6, **b)** r^{12}, **c)** $1/r^{12}$, **d)** $1/r^6$, **e)** $1/r^6$, **f)** $1/r^{12}$, **g)** r^{12}, **h)** r^6

3.17. y^{b+d}/x^{a+c} **3.18.** p^{w+y}/q^{x+z} **3.19.** mnb/a **3.20.** xyu^2/v^3

3.21. $\left(36s^{30}\right)/\left(25r^{16}\right)$ **3.22.** $\left(-125x^{60}\right)/\left(8y^{36}\right)$ **3.23.** $\left(-8p^{18}\right)/\left(27q^{21}\right)$

3.24. $\left(9m^{20}\right)/\left(25n^8\right)$

Session 4

4.1. $\dfrac{6y-4}{5}$ **4.2.** $\dfrac{12z-8}{7}$ **4.3.** $\dfrac{z(3z+5)}{2(z-3)}$ **4.4.** $\dfrac{2y(2y+3)}{3y-5}$

4.5. $\dfrac{2(x+5)}{3(x+4)}$ **4.6.** $\dfrac{3(x-2)}{4(x-1)}$ **4.7.** $\dfrac{2(x+4)}{7x}$ **4.8.** $\dfrac{3(x+3)}{4x}$

Session 4 (*continued*)

4.9. 2

4.10. 2

4.11. $\dfrac{10m+15}{36}$

4.12. $\dfrac{7n+50}{60}$

4.13. $\dfrac{14x+15y}{24xy}$

4.14. $\dfrac{9x+28y}{24xy}$

4.15. $\dfrac{35x^2-50x+8}{30x^2}$

4.16. $\dfrac{9x^2-28x+10}{24x^2}$

4.17. $\dfrac{x+23}{(4x-7)(7x-4)}$

4.18. $\dfrac{3x+10}{(3x-4)(4x-3)}$

4.19. $\dfrac{2a}{a-4}$

4.20. $\dfrac{2b}{b-5}$

4.21. $\dfrac{3x}{x-2}$

4.22. $\dfrac{2x}{x-4}$

4.23. $\dfrac{x+3}{5(x-5)}$

4.24. $\dfrac{x+5}{4(x-4)}$

4.25. $\dfrac{2}{(x+3)(x+5)}$

4.26. $\dfrac{1}{(x+4)(x+5)}$

4.27. $-\dfrac{13}{(b+4)(b-2)}$

4.28. $-\dfrac{25}{(c-3)(c+5)}$

4.29. $\dfrac{4d}{9c^2}$

4.30. $\dfrac{15n}{m^3}$

4.31. $\dfrac{3x+2}{4-5x}$

4.32. $\dfrac{2x-3}{5+4x}$

4.33. $\dfrac{14x}{3}$

4.34. $\dfrac{15x}{4}$

4.35. $-\dfrac{7}{24}$

4.36. $\dfrac{41}{2}$

4.37. $\dfrac{4x+3xy}{6y-2x^2}$

4.38. $\dfrac{6xy-5y}{4y^2+3x}$

4.39. -3

4.40. -4

4.41. $\dfrac{4k-19}{6k-13}$

4.42. $\dfrac{5m-12}{7m-34}$

4.43. $\dfrac{x-1}{x+4}$

4.44. $\dfrac{x-1}{x+2}$

4.45. $\dfrac{5x+8}{5x-8}$

4.46. $\dfrac{7x+6}{7x-4}$

4.47. $\dfrac{(-a-b)x-ac}{(e-d)x+ce}$

4.48. $\dfrac{(-a+c)x-bc}{(-d-e)x+bd}$

Session 5

5.1. $x=9/7$

5.2. $x=5$

5.3. $x=4/9$

5.4. $x=-6/5$

5.5. $x=17/3$

5.6. $x=7$

5.7. $x=23$

5.8. $x=17$

5.9. $x=6/5$

5.10. $x=25/2$

5.11. $x=2/9$

5.12. $x=1/6$

5.13. $x=-1/12$

5.14 $x=-11/5$

5.15. $x=-11$

5.16. $x=-6$

5.17. $m=-113/11$

5.18. $m=-15$

5.19. $x=3$

5.20. $x=-11$

5.25. 2 gallons

5.26. 4 gallons

5.27. 36%

5.28. 14%

Session 6

6.1. **a)** 4   **b)** 3   **c)** 25   **d)** 4   **e)** 1/2   **f)** 125   **g)** 1/9   **h)** 2

6.2. **a)** 9   **b)** 4   **c)** 36   **d)** 3   **e)** 1/6   **f)** 4   **g)** 1/8   **h)** 3

6.3. $\dfrac{1}{\sqrt{a}}$ **6.4.** $\sqrt{b}$ **6.5.** $5\sqrt{2}$ **6.6.** $3\sqrt{6}$

6.7. $4\sqrt{2}$ **6.8.** $6\sqrt{2}$ **6.9.** **a)** x^4 **6.10.** **a)** x^3

                                                   b) $x^3\sqrt{x}$ **b)** $x^2\sqrt{x}$

6.11. **a)** $6y^{18}$ **6.12.** **a)** $7z^{24}\sqrt{z}$ **6.13.** $15x^5y^2z^8\sqrt{3y}$ **6.14.** $25x^7y^4z^6\sqrt{3yz}$

       b) $3y^4\sqrt{y}$ **b)** $8z^{32}$

6.15. $\left(b^{16}\right)/\left(a^9\right)$ **6.16.** $\left(a^9\right)/\left(b^{25}\right)$

Session 7

7.1. a) 7   **b)** 2018   **c)** $47\sqrt{2}$ **7.2. a)** 5   **b)** 2019   **c)** $26\sqrt{3}$ **7.3.** $7\sqrt{6}$ **7.4.** $6\sqrt{15}$

7.5. a) $3p^4q^4\sqrt{5q}$ **b)** $576a^5b\sqrt{3b}$ **7.6. a)** $5m^5n^4\sqrt{3m}$ **b)** $294u^6v^3\sqrt{2v}$

7.7. a) $11\sqrt{6}$ **b)** $-2\sqrt{7}$ **7.8. a)** $9\sqrt{5}$ **b)** $-3\sqrt{3}$ **7.9.** $8m\sqrt{6n}+2\sqrt{7k}$

7.10. $12p\sqrt{3q}+2\sqrt{6r}$ **7.11. a)** $7\sqrt{2}$ **b)** $7\sqrt{3}$ **c)** $10\sqrt{6}-10\sqrt{5}$

7.12. a) $5\sqrt{3}$ **b)** $2\sqrt{5}$ **c)** $28\sqrt{7}-40\sqrt{3}$ **7.13.** $2\sqrt{5}+5\sqrt{2}$ **7.14.** $2\sqrt{3}+3\sqrt{2}$

7.15. 315   **7.16.** 24   **7.17.** $4+8\sqrt{30}$ **7.18.** $-26-2\sqrt{14}$ **7.19.** 42   **7.20.** 67

7.21. 117   **7.22.** 189   **7.23.** $83+12\sqrt{35}$ **7.24.** $72+48\sqrt{2}$ **7.25.** $83-12\sqrt{35}$

7.26. $72-48\sqrt{2}$

Session 8

8.1. $\dfrac{5\sqrt{6}}{6}$ **8.2.** $\dfrac{4\sqrt{3}}{3}$ **8.3.** $\dfrac{\sqrt{15}}{5}$ **8.4.** $\dfrac{\sqrt{42}}{7}$

8.5. $\dfrac{\sqrt{5}}{15}$ **8.6.** $\dfrac{\sqrt{3}}{15}$ **8.7.** $\dfrac{4\sqrt{3}}{15}$ **8.8.** $\dfrac{2\sqrt{5}}{15}$

8.9. $\dfrac{5-\sqrt{3}}{4}$ **8.10.** $\dfrac{3-\sqrt{5}}{2}$ **8.11.** $2+\sqrt{3}$ **8.12.** $\dfrac{5-\sqrt{6}}{19}$

Session 8 (*continued*)

8.13. $\dfrac{\sqrt{6}-\sqrt{3}}{3}$

8.14. $\sqrt{6}+\sqrt{5}$

8.15. $\dfrac{4u+3\sqrt{uv}+8\sqrt{u}+6\sqrt{v}}{16u-9v}$

8.16. $\dfrac{2a-3\sqrt{ab}-5b}{4a-25b}$

8.17. $x=7$

8.18. $x=13$

8.19. $x=8$

8.20. $x=5$

8.21. No solutions

8.22. No solutions

8.23. $x=4$

8.24. $x=3$

8.25. No solutions

8.26. No solutions

8.27. a) $x=2$ **b)** $x=-2$ and $x=-1$ **8.28. a)** $x=-3$ and $x=-1$ **b)** $x=1$

8.29. a) $x=1$ and $x=3$ **b)** $x=5$ **8.30. a)** $x=2$ and $x=4$ **b)** $x=7$

Session 9

9.1. a) $5i$

b) $4\sqrt{2}i$

9.2. a) $4i$

b) $3\sqrt{3}i$

9.3. a) $9-2i$

b) $-4-8i$

9.4. a) $12-5i$

b) $4+5i$

9.5. a) $6-12i$ **b)** $-8+20i$ **c)** $-7+22i$ **d)** $-12-54i$ **e)** $45-7i$ **f)** 13

9.6. a) $12+20i$ **b)** $21+42i$ **c)** $4+38i$ **d)** $17-57i$ **e)** $34-38i$ **f)** 20

9.7. a) $\dfrac{8}{5}+\dfrac{2}{5}i$ **b)** $-\dfrac{3}{2}i$ **c)** $\dfrac{3}{2}+\dfrac{9}{4}i$ **d)** $-\dfrac{38}{61}-\dfrac{9}{61}i$ **e)** $\dfrac{2}{5}-\dfrac{9}{5}i$ **f)** $\dfrac{5}{13}-\dfrac{12}{13}i$

9.8. a) $-3+\dfrac{7}{2}i$ **b)** $\dfrac{2}{3}i$ **c)** $\dfrac{1}{2}-\dfrac{2}{3}i$ **d)** $\dfrac{38}{25}-\dfrac{9}{25}i$ **e)** $-\dfrac{2}{13}-\dfrac{23}{13}i$ **f)** $-\dfrac{45}{53}-\dfrac{28}{53}i$

9.9. a) -1 **b)** $-i$ **c)** 1 **d)** i

9.10. a) $-i$ **b)** 1 **c)** i **d)** -1

9.11. a) $\left\{\dfrac{3\pm\sqrt{6}}{4}\right\}$ **b)** $\left\{\pm\sqrt{7}i\right\}$ **c)** $\left\{-\dfrac{2}{3}\pm\dfrac{\sqrt{3}}{6}i\right\}$

9.12. a) $\left\{\dfrac{-2\pm\sqrt{3}}{5}\right\}$ **b)** $\left\{\pm\sqrt{3}i\right\}$ **c)** $\left\{\dfrac{5}{7}\pm\dfrac{\sqrt{6}}{7}i\right\}$

Session 10

10.1. a) $x^2 + 8x + 16 = (x+4)^2$ **b)** $x^2 - 3x + \dfrac{9}{4} = \left(x - \dfrac{3}{2}\right)^2$

    c) $x^2 + x + \dfrac{1}{4} = \left(x + \dfrac{1}{2}\right)^2$ **d)** $x^2 - \dfrac{5}{4}x + \dfrac{25}{64} = \left(x - \dfrac{5}{8}\right)^2$

10.2. a) $x^2 + 6x + 9 = (x+3)^2$ **b)** $x^2 - 9x + \dfrac{81}{4} = \left(x - \dfrac{9}{2}\right)^2$

    c) $x^2 - x + \dfrac{1}{4} = \left(x - \dfrac{1}{2}\right)^2$ **d)** $x^2 + \dfrac{7}{3}x + \dfrac{49}{36} = \left(x + \dfrac{7}{6}\right)^2$

10.3. a) $\{-1,\, 5\}$ **b)** $\{-3,\, -2\}$ **c)** $\left\{\dfrac{7 \pm \sqrt{17}}{2}\right\}$ **d)** $\{-4 \pm \sqrt{5}i\}$ **e)** $\left\{-3,\, \dfrac{3}{2}\right\}$

10.4. a) $\{-4,\, 2\}$ **b)** $\{2,\, 7\}$ **c)** $\left\{\dfrac{-3 \pm \sqrt{29}}{2}\right\}$ **d)** $\{5 \pm \sqrt{3}i\}$ **e)** $\left\{-2,\, -\dfrac{1}{3}\right\}$

10.5. a) No real roots   **b)** Two roots   **c)** One root

10.6. a) One root   **b)** No real roots   **c)** Two roots

10.7. a) $\left\{\dfrac{4 \pm \sqrt{21}}{5}\right\}$ **b)** $\left\{-\dfrac{3}{4}\right\}$ **c)** $\left\{\dfrac{7}{8} \pm \dfrac{\sqrt{31}}{8}i\right\}$ **d)** $\left\{\dfrac{4}{3},\, \dfrac{3}{2}\right\}$

10.8. a) $\left\{\dfrac{3 \pm \sqrt{30}}{7}\right\}$ **b)** $\left\{\dfrac{7}{3}\right\}$ **c)** $\left\{-\dfrac{5}{12} \pm \dfrac{\sqrt{47}}{12}i\right\}$ **d)** $\left\{\dfrac{5}{6},\, \dfrac{3}{2}\right\}$

Session 11

11.1. a) $(-4,\, -5)$, Up   **b)** $(2,\, 6)$, Down   **c)** $(5,\, -4)$, Up

11.2. a) $(3,\, 7)$, Down   **b)** $(-7,\, -3)$, Up   **c)** $(-1,\, 8)$, Down

11.3. a) squared form: $y = 3(x+2)^2 - 3$, standard form: $y = 3x^2 + 12x + 9$

    b) squared form: $y = 3(x+2)^2 + 1$, standard form: $y = 3x^2 + 12x + 13$

11.4. a) squared form: $y = 3(x-2)^2 + 1$, standard form: $y = 3x^2 - 12x + 13$

    b) squared form: $y = 3(x-2)^2 - 3$, standard form: $y = 3x^2 - 12x + 9$

11.5. a) squared form: $y = -2(x-3)^2 + 2$, standard form: $y = -2x^2 + 12x - 16$

Session 11 (*continued*)

b) squared form: $y = -2(x-3)^2 - 1$, standard form: $y = -2x^2 + 12x - 19$

11.6. a) squared form: $y = -2(x+3)^2 - 1$, standard form: $y = -2x^2 - 12x - 19$

b) squared form: $y = -2(x+3)^2 + 2$, standard form: $y = -2x^2 - 12x - 16$

11.7. a) vertex: $(1, -4)$, *y*- intercepts: $(0, -3)$, *x*-intercepts: $(-1, 0)$ and $(3, 0)$

b) vertex: $(-2, 9)$, *y*- intercepts: $(0, 5)$, *x*-intercepts: $(-5, 0)$ and $(1, 0)$

11.8. a) vertex: $(-1, 9)$, *y*- intercepts: $(0, 8)$, *x*-intercepts: $(-4, 0)$ and $(2, 0)$

b) vertex: $(3, -4)$, *y*- intercepts: $(0, 5)$, *x*-intercepts: $(1, 0)$ and $(5, 0)$

Session 12

12.1. $2\sqrt{10}$ **12.2.** $5\sqrt{2}$

12.3. Acute triangle. Biggest angle is *C*, smallest angle is *A*

12.4. Right triangle. Biggest angle is *A*, smallest angle is *C*

12.5. $(-1, -5)$ **12.6.** $(3, 1)$ **12.7.** $(1, 3)$ **12.8.** $(-7, -2)$

12.9. a) Center $(-2, -4)$, radius $= 6$ **b)** Center $(-5, 2)$,   radius $= 2\sqrt{5}$

12.10. a) Center $(6, 3)$,   radius $= 7$ **b)** Center $(7, -8)$, radius $= 5\sqrt{2}$

12.11. $(x+3)^2 + (y+4)^2 = 13^2$ **12.12.** $(x-12)^2 + (y-10)^2 = 17^2$

12.13. a) Center $(-2, 1)$,   radius $= 4$, points: $(-2, 5)$, $(-2, -3)$, $(-6, 1)$, $(2, 1)$

b) Center $(5, 4)$,   radius $= 3$, points: $(5, 1)$, $(5, 7)$, $(2, 4)$, $(8, 4)$

12.14. a) Center $(4, -1)$,   radius $= 3$, points: $(4, 2)$, $(4, -4)$, $(1, -1)$, $(7, -1)$

b) Center $(-3, -2)$, radius $= 5$, points: $(-3, -7)$, $(-3, 3)$, $(-8, -2)$, $(2, -2)$

Session 13

13.1. a) $\{(-7, -22), (1, 2)\}$ **13.2. a)** $\{(-1, 6), (13, -50)\}$

b) $\{(-8, -3)\}$ **b)** $\{(70, 6)\}$

c) $\{(9, -3)\}$ **c)** $\{(4, 2)\}$

d) $\{(-19, -11), (1, -1)\}$ **d)** $\{(-34, 12), (2, 0)\}$

e) $\{(3, 4), (-3, 4), (3, -4), (-3, -4)\}$ **e)** $\{(2, 4), (-2, 4), (2, -4), (-2, -4)\}$

Session 13 (*continued*)

13.1. f) $\{(1, 3), (-1, 3), (3, -1), (-3, -1)\}$

13.2. f) $\{(2, 5), (-2, 5), (2, -5), (-2, -5)\}$

13.3. Length = 4 m, width = 3 m   **13.4.** Length = 6 yd, width = 5 yd

Session 14

14.1. a) $50° + 360°n,\ n = 0, \pm 1, \pm 2,...$ **b)** $-70° + 360°n,\ n = 0, \pm 1, \pm 2,...$

$410°,\ 770°,\ -310°,\ -670°$ $290°,\ 650°,\ -430°,\ -790°$

14.2. a) $27° + 360°n,\ n = 0, \pm 1, \pm 2,...$ **b)** $-35° + 360°n,\ n = 0, \pm 1, \pm 2,...$

$387°,\ 747°,\ -333°,\ -693°$ $325°,\ 685°,\ -395°,\ -755°$

14.3. a) $b = 6\sqrt{3},\ c = 12$ **b)** $a = \sqrt{3},\ c = 2\sqrt{3}$ **c)** $a = 4,\ b = 4\sqrt{3}$

14.4. a) $b = 8\sqrt{3},\ c = 16$ **b)** $a = 3\sqrt{3},\ c = 6\sqrt{3}$ **c)** $a = 2,\ b = 2\sqrt{3}$

14.5. a) $b = 4,\ c = 4\sqrt{2}$ **b)** $a = 8,\ c = 8\sqrt{2}$ **c)** $a = b = 9\sqrt{2}/2$

14.6. a) $b = 3,\ c = 3\sqrt{2}$ **b)** $a = 6,\ c = 6\sqrt{2}$ **c)** $a = b = 7\sqrt{2}/2$

14.7. Side opposite to $30°$ is $c/2$, side opposite to $60°$ is $c\sqrt{3}/2$

14.8. Side opposite to $30°$ is $b\sqrt{3}/3$, hypotenuse is $2b\sqrt{3}/3$

14.9. Both sides are $c\sqrt{2}/2$

Session 15

15.1. $\sin A = \dfrac{3}{5},\ \cos A = \dfrac{4}{5},\ \tan A = \dfrac{3}{4},\ \sin B = \dfrac{4}{5},\ \cos B = \dfrac{3}{5},\ \tan B = \dfrac{4}{3}.$

15.2. $\sin A = \dfrac{12}{13},\ \cos A = \dfrac{5}{13},\ \tan A = \dfrac{12}{5},\ \sin B = \dfrac{5}{13},\ \cos B = \dfrac{12}{13},\ \tan B = \dfrac{5}{12}.$

15.3. a) $\cos\theta = \dfrac{12}{3},\ \tan\theta = \dfrac{5}{12},\ \csc\theta = \dfrac{13}{5},\ \sec\theta = \dfrac{3}{12},\ \cot\theta = \dfrac{12}{5}.$

   b) $\sin\theta = \dfrac{\sqrt{13}}{7},\ \tan\theta = \dfrac{\sqrt{13}}{6},\ \sec\theta = \dfrac{7}{6},\ \csc\theta = \dfrac{7\sqrt{13}}{13},\ \cot\theta = \dfrac{6\sqrt{13}}{13}.$

   c) $\sin\theta = \dfrac{4\sqrt{41}}{41},\ \cos\theta = \dfrac{5\sqrt{41}}{41},\ \cot\theta = \dfrac{4}{5},\ \csc\theta = \dfrac{\sqrt{41}}{4},\ \sec\theta = \dfrac{\sqrt{41}}{5}.$

Session 15 (*continued*)

15.4. a) $\cos\theta = \dfrac{4}{5}$, $\tan\theta = \dfrac{3}{4}$, $\csc\theta = \dfrac{5}{3}$, $\sec\theta = \dfrac{5}{4}$, $\cot\theta = \dfrac{4}{3}$.

b) $\sin\theta = \dfrac{\sqrt{11}}{6}$, $\tan\theta = \dfrac{\sqrt{11}}{5}$, $\sec\theta = \dfrac{6}{5}$, $\csc\theta = \dfrac{6\sqrt{11}}{11}$, $\cot\theta = \dfrac{5\sqrt{11}}{11}$.

c) $\sin\theta = \dfrac{3\sqrt{13}}{13}$, $\cos\theta = \dfrac{2\sqrt{13}}{13}$, $\cot\theta = \dfrac{2}{3}$, $\csc\theta = \dfrac{\sqrt{13}}{3}$, $\sec\theta = \dfrac{\sqrt{13}}{2}$.

15.5. $C = 58^0$
$AB = 43.209$
$BC = 50.951$

15.6. $A = 37^0$
$AB = 58.850$
$BC = 35.417$

15.7. $B = 67^0$
$AB = 48.886$
$BC = 19.101$

15.8. $C = 54^0$
$AC = 66.748$
$BC = 39.233$

15.9. $A = 27^0$
$AB = 16.038$
$BC = 8.172$

15.10. $C = 48^0$
$AB = 17.092$
$AC = 15.390$

15.11. $A = 32^0$
$B = 58^0$
$AB = 9.434$

15.12. $A = 66^0$
$B = 24^0$
$AB = 9.849$

15.13. $A = 51^0$
$B = 39^0$
$AC = 5.657$

15.14. $B = 33^0$
$C = 57^0$
$AB = 9.220$

15.15. $\sec 30° = \dfrac{2\sqrt{3}}{3}$, $\csc 30° = 2$, $\sec 45° = \csc 45° = \sqrt{2}$, $\sec 60° = 2$, $\csc 60° = \dfrac{2\sqrt{3}}{3}$.

15.16. $\cot 30° = \sqrt{3}$, $\cot 45° = 1$, $\cot 60° = \dfrac{\sqrt{3}}{3}$.

15.17. B

15.18. A

15.19. A

15.20. B

15.21. 29.1 ft

15.22. 61 ft

15.23. 11.5 ft

15.24. 2.5 m

15.25. 167.1 ft

15.26. 83.1 ft

15.27. 3.1°

15.28. 29.1°

15.29. 46.1°

15.30. 21.1°

15.31. 63.9°

15.32. 53.1°

15.33. 20.3 m

15.34. 60^0

Session 16

16.1. a) $-\sin\theta$ **b)** $\sin\theta$ **16.2. a)** $-\cos\theta$ **b)** $-\cos\theta$

16.3. a) 50°   **b)** 40°   **c)** 70°   **d)** 85°   **e)** 50°

16.4. a) 20°   **b)** 10°   **c)** 50°   **d)** 80°   **e)** 30°

16.5. a) 220°   **b)** 110°   **c)** 310°   **d)** 20°

16.6. a) 320°   **b)** 70°   **c)** 130°   **d)** 200°

16.7. a) III, 30°, $-\dfrac{1}{2}$ **b)** IV, 60°, $\dfrac{1}{2}$ **c)** II, 45°, -1

16.8. a) IV, 45°, $-\dfrac{\sqrt{2}}{2}$ **b)** II, 30°, $-\dfrac{\sqrt{3}}{2}$ **c)** III, 60°, $\sqrt{3}$

Session 16 (*continued*)

16.9.

a) IV, $\cos\theta = \dfrac{\sqrt{5}}{3}$, $\tan\theta = -\dfrac{2\sqrt{5}}{5}$, $\csc\theta = -\dfrac{3}{2}$, $\sec\theta = \dfrac{3\sqrt{5}}{5}$, $\cot\theta = -\dfrac{\sqrt{5}}{2}$

b) II, $\sin\theta = \dfrac{\sqrt{21}}{5}$, $\tan\theta = -\dfrac{\sqrt{21}}{2}$, $\csc\theta = \dfrac{5\sqrt{21}}{21}$, $\sec\theta = -\dfrac{5}{2}$, $\cot\theta = -\dfrac{2\sqrt{21}}{21}$

c) III, $\sin\theta = -\dfrac{3\sqrt{34}}{34}$, $\cos\theta = -\dfrac{5\sqrt{34}}{34}$, $\csc\theta = -\dfrac{\sqrt{34}}{3}$, $\sec\theta = -\dfrac{\sqrt{34}}{5}$, $\cot\theta = \dfrac{5}{3}$

16.10.

a) II, $\cos\theta = -\dfrac{\sqrt{33}}{7}$, $\tan\theta = -\dfrac{4\sqrt{33}}{33}$, $\csc\theta = \dfrac{7}{4}$, $\sec\theta = -\dfrac{7\sqrt{33}}{33}$, $\cot\theta = -\dfrac{\sqrt{33}}{4}$

b) III, $\sin\theta = -\dfrac{\sqrt{39}}{8}$, $\tan\theta = \dfrac{\sqrt{39}}{5}$, $\csc\theta = -\dfrac{8\sqrt{39}}{39}$, $\sec\theta = -\dfrac{8}{5}$, $\cot\theta = \dfrac{5\sqrt{39}}{39}$

c) IV, $\sin\theta = -\dfrac{7\sqrt{65}}{65}$, $\cos\theta = \dfrac{4\sqrt{65}}{65}$, $\csc\theta = -\dfrac{\sqrt{65}}{7}$, $\sec\theta = \dfrac{\sqrt{65}}{4}$, $\cot\theta = -\dfrac{4}{7}$

16.11.

a) $\sin\theta = -\dfrac{2\sqrt{5}}{5}$, $\cos\theta = -\dfrac{\sqrt{5}}{5}$, $\tan\theta = 2$, $\csc\theta = -\dfrac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$, $\cot\theta = \dfrac{1}{2}$

b) $\sin\theta = -\dfrac{5\sqrt{41}}{41}$, $\cos\theta = \dfrac{4\sqrt{41}}{41}$, $\tan\theta = -\dfrac{5}{4}$, $\csc\theta = -\dfrac{\sqrt{41}}{5}$, $\sec\theta = \dfrac{\sqrt{41}}{4}$, $\cot\theta = -\dfrac{4}{5}$

c) $\sin\theta = \dfrac{7\sqrt{58}}{58}$, $\cos\theta = -\dfrac{3\sqrt{58}}{58}$, $\tan\theta = -\dfrac{7}{3}$, $\csc\theta = \dfrac{\sqrt{58}}{7}$, $\sec\theta = -\dfrac{\sqrt{58}}{3}$, $\cot\theta = -\dfrac{3}{7}$

16.12.

a) $\sin\theta = \dfrac{5\sqrt{34}}{34}$, $\cos\theta = -\dfrac{3\sqrt{34}}{34}$, $\tan\theta = -\dfrac{5}{3}$, $\csc\theta = \dfrac{\sqrt{34}}{5}$, $\sec\theta = -\dfrac{\sqrt{34}}{3}$, $\cot\theta = -\dfrac{3}{5}$

b) $\sin\theta = \dfrac{6\sqrt{61}}{61}$, $\cos\theta = -\dfrac{5\sqrt{61}}{61}$, $\tan\theta = -\dfrac{6}{5}$, $\csc\theta = \dfrac{\sqrt{61}}{6}$, $\sec\theta = -\dfrac{\sqrt{61}}{5}$, $\cot\theta = -\dfrac{5}{6}$

c) $\sin\theta = -\dfrac{7\sqrt{65}}{65}$, $\cos\theta = -\dfrac{4\sqrt{65}}{65}$, $\tan\theta = \dfrac{7}{4}$, $\csc\theta = -\dfrac{\sqrt{65}}{7}$, $\sec\theta = -\dfrac{\sqrt{65}}{4}$, $\cot\theta = \dfrac{4}{7}$

Session 17

17.1. $p = 105.3$, $q = 129.1$, $R = 20°$

17.2. $q = 10.8$, $r = 21.3$, $P = 85°$

17.3. $l = 21.2$, $K = 25.4°$, $L = 114.6°$

17.4. $k = 36.4$, $K = 77.5°$, $M = 32.5°$

Session 17 (*continued*)

17.5. 9.1 m

17.6. 21.9 m

17.7. $AB = 33$ m, $BC = 26.9$ m

17.8. 11.1 ft and 12.0 ft.

17.9. 39.9°

17.10. 38.0°

17.11. 98.6°

17.12. 112.3°

17.13. 11.2 m

17.14. 32.5 m

17.23. 3.8 mi

17.24. $\dfrac{d\sin A\sin B}{\sin(A+B)}$

17.25. 6.3 mi

17.26. $\dfrac{d\sin A\sin B}{\sin|A-B|}$

17.27. 1.3 m and 2.2 m

Session 18

18.1. $p = 28.2$, $Q = 40.1°$, $R = 74.9°$

18.2. $q = 96.3$, $P = 16.2°$, $R = 33.8°$

18.3. $K = 28.9°$, $L = 46.6°$, $M = 104.5°$

18.4. $K = 133.4°$, $L = 29.0°$, $M = 17.6°$

18.5. Yes

18.6. 301.4 ft

18.7. 41.8°

18.8. 29.3°

18.9. 14°

18.10. 57°

18.11. 11.5°

18.12. 280 m

Session 19

19.1. a) 74.5°, **b)** −34.4°

19.2. a) 137.5°, **b)** −45.8°

19.3. a) 40° **b)** −54°

19.4. a) 48°, **b)** −216°

19.5. a) 2.44, **b)** − 1.48

19.6. a) 1.36, **b)** − 4.14

19.7. a) $\dfrac{2\pi}{3}$, **b)** $-\dfrac{5\pi}{6}$

19.8. a) $\dfrac{11\pi}{6}$, **b)** $-\dfrac{5\pi}{4}$

19.9. a) $\sin\left(\dfrac{4\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$, $\cos\left(\dfrac{4\pi}{3}\right) = -\dfrac{1}{2}$, $\tan\left(\dfrac{4\pi}{3}\right) = \sqrt{3}$

b) $\sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$, $\cos\left(-\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$, $\tan\left(-\dfrac{\pi}{4}\right) = -1$

c) $\sin\left(\dfrac{5\pi}{6}\right) = \dfrac{1}{2}$, $\cos\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$, $\tan\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{3}$

19.10. a) $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$, $\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$, $\tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{3}$

b) $\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$, $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$, $\tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$

Session 19 (*continued*)

c) $\sin\left(-\dfrac{3\pi}{4}\right)=-\dfrac{\sqrt{2}}{2}$, $\cos\left(-\dfrac{3\pi}{4}\right)=-\dfrac{\sqrt{2}}{2}$, $\tan\left(-\dfrac{3\pi}{4}\right)=1$

19.11. a) 9.7 ft **b)** 1.8 cm   **19.12. a)** 0.5 m **b)** 5.0 in   **19.13.** $\dfrac{\pi}{4}$, $\dfrac{\pi}{3}$, $\dfrac{5\pi}{12}$

19.14. 16 cabs, arc $\approx$ 3.14 m   **19.15.** 180 m/min   **19.16.** 3 sec

Session 20

20.1. a) $\dfrac{\pi}{4}$, $\dfrac{3\pi}{4}$, **b)** $\dfrac{4\pi}{3}$, $\dfrac{5\pi}{3}$, **c)** Ø (No solutions)

20.2. a) $\dfrac{\pi}{3}$, $\dfrac{2\pi}{3}$, **b)** $\dfrac{7\pi}{6}$, $\dfrac{11\pi}{6}$, **c)** Ø (No solutions)

20.3. a) $0, \pi, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$, **b)** $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$ **20.4. a)** $0, \pi, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$, **b)** $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{2}$

20.5. a) 0.85, 2.29 **b)** 3.48, 5.94   **20.6. a)** 0.64, 2.50 **b)** 3.75, 5.67

Session 21

21.1. a) $\dfrac{\pi}{4}$, $\dfrac{7\pi}{4}$, **b)** $\dfrac{5\pi}{6}$, $\dfrac{7\pi}{6}$ **c)** Ø **21.2. a)** $\dfrac{\pi}{6}$, $\dfrac{11\pi}{6}$ **b)** $\dfrac{3\pi}{4}$, $\dfrac{5\pi}{4}$, **c)** Ø

21.3. a) $\dfrac{\pi}{6}$, $\dfrac{7\pi}{6}$, **b)** $\dfrac{2\pi}{3}$, $\dfrac{5\pi}{3}$ **21.4. a)** $\dfrac{\pi}{3}$, $\dfrac{4\pi}{3}$ **b)** $\dfrac{5\pi}{3}$, $\dfrac{11\pi}{3}$

21.5. a) $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ **b)** $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ **c)** $0, \pi, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

21.6. a) $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ **b)** $\pi, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ **c)** $0, \pi, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

21.7. a) 0.93, 5.35 **b)** 2.18, 4.10   **21.8. a)** 0.72, 5.56 **b)** 1.91, 4.37

21.9. a) 1.11, 4.25 **b)** 2.68, 5.82   **21.10. a)** 1.33, 4.47 **b)** 2.55, 5.70

Session 22

22.13. $0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ **22.14.** $0, \dfrac{\pi}{2}, \pi$ **22.15.** $0, \pi, \dfrac{3\pi}{2}$

22.16. $\dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ **22.17.** $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ **22.18.** $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

22.19. $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ **22.20.** $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ **22.21.** $\pi, 0.841, 5.442$

Session 22 (*continued*)

22.22. $\dfrac{\pi}{2}$, 4.069, 5.356   **22.23.** 0, $\dfrac{\pi}{6}$, π, $\dfrac{7\pi}{6}$ **22.24.** $\dfrac{\pi}{3}$, $\dfrac{\pi}{2}$, $\dfrac{4\pi}{3}$, $\dfrac{3\pi}{2}$

22.25. $\dfrac{\pi}{2}$, π **22.26.** 0, $\dfrac{3\pi}{2}$

Session 23

23.1. a) $2^4 = 16$ **b)** $2^{-4} = \dfrac{1}{16}$ **23.2. a)** $4^3 = 64$ **b)** $4^{-3} = \dfrac{1}{64}$

23.3. a) $\log_2 8 = 3$ **b)** $\log_2 \dfrac{1}{8} = -3$ **23.4. a)** $\log_3 81 = 4$ **b)** $\log_3 \dfrac{1}{81} = -4$

23.5. a) 4   **b)** 3   **c)** -2 **d)** -6 **e)** -2 **f)** $\dfrac{1}{5}$ **g)** $\dfrac{9}{4}$

23.6. a) 2   **b)** 4   **c)** -3 **d)** -3 **e)** -2 **f)** $\dfrac{1}{4}$ **g)** $\dfrac{11}{5}$

23.7. a) 3   **b)** $-\dfrac{2}{3}$ **c)** -3 **23.8. a)** 4   **b)** $\dfrac{1}{2}$ **c)** -2

23.9. a) 7   **b)** 4   **23.10. a)** 4   **b)** 3   **23.11.** 1   **23.12.** 3   **23.13.** $3u + 2v$

23.14. $4u + 5v$ **23.15.** $7u - 6v$ **23.16.** $3v - 2u$

23.17. a) $\log\left(\dfrac{x^3 \sqrt[3]{z^2}}{y^2}\right)$ **b)** $\log\left(\dfrac{x^5 y^6}{\sqrt[4]{z^3}}\right)$ **23.18. a)** $\log\left(\dfrac{x^4 \sqrt[5]{z^4}}{y^3}\right)$ **b)** $\log\left(\dfrac{x^6 y^7}{\sqrt[6]{z^5}}\right)$

23.19. a) 1.226   **b)** -0.429 **c)** -0.348 **23.20. a)** 1.292   **b)** 0.881   **c)** 1.931

23.21. 1.069   **23.22.** 0.936

Session 24

24.1. $f \leftrightarrow B$, $g \leftrightarrow A$ **24.2.** $f \leftrightarrow A$, $g \leftrightarrow B$ **24.3.** $f \leftrightarrow A$, $g \leftrightarrow B$

24.4. $f \leftrightarrow B$, $g \leftrightarrow A$ **24.5.** $f \leftrightarrow B$, $g \leftrightarrow A$ **24.6.** $f \leftrightarrow A$, $g \leftrightarrow B$

24.7. $f \leftrightarrow A$, $g \leftrightarrow B$ **24.8.** $f \leftrightarrow B$, $g \leftrightarrow A$ **24.9. a)** $\log_3(x)$ **b)** $\left(\dfrac{2}{3}\right)^x$

24.10. a) $\log_{\frac{3}{4}}(x)$ **b)** 5^x **24.11. a)** $\left(-3/2, \infty\right)$ **b)** $\left(-\infty, 3/2\right)$

24.12. a) $\left(-5/4, \infty\right)$ **b)** $\left(-\infty, 5/4\right)$

Session 25

25.1. $I = \$7.29$, $A = \$707.29$

25.2. $I = \$14.00$, $A = \$614.00$

25.3. $A = \$1,540.03$, $I = \$340.03$

25.4. $A = \$2,704.38$, $I = \$304.38$

25.5. $A = \$1,540.83$, $I = \$340.83$

25.6. $A = \$2,705.99$, $I = \$305.99$

25.7. 17.4 years

25.8. 9.9 years

29.9. Bank *B*. Yes, information about investment amount and time is not needed.