KEY

① HOMEWORK 4, p.329

$P(n)$ is the statement that $1^3+2^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2$ for the positive integer $n$.

a) What is the statement $P(1)$?

$P(1)$ is the statement $1^3=\left[\frac{1(1+1)}{2}\right]^2$

b) Show that $P(1)$ is true, completing the basis step of the proof.

$1^3=1$ and $\left[\frac{1(1+1)}{2}\right]^2=1$.

$P(1)$ is true, because both sides of $P(1)$ are the same.

c) What is the inductive hypothesis?

The inductive hypothesis is $P(K)$ which is the statement that

$1^3+2^3+\cdots+K^3=\left[\frac{K(K+1)}{2}\right]^2$

d) What do you need to prove in the inductive step?

We need to show that $P(K)$ implies $P(K+1)$, where $K$ is a positive integer.

In other words we need to show that the inductive hypothesis $P(K)$ implies that

$1^3+2^3+\cdots+K^3+(K+1)^3=\left[\frac{(K+1)(K+2)}{2}\right]^2$

e) Complete the inductive step, identifying where you use the inductive hypothesis.

We have $\underbrace{\boxed{1^3+2^3+\cdots+K^3}+(K+1)^3=\boxed{\left[\frac{K(K+1)}{2}\right]^2}}_{=\text{ by inductive hypothesis}}+(K+1)^3 \underset{\text{algebra}}{=}$

$$\frac{K^2(K+1)^2}{4}+\frac{4(K+1)^3}{4}=\frac{K^2(K+1)^2+4(K+1)^3}{4}\underset{\text{factor out }(K+1)^2}{=}\frac{(K+1)^2[K^2+4(K+1)]}{4}=\frac{(K+1)^2[K^2+4K+4]}{4}$$

$$=\frac{(K+1)^2(K+2)^2}{4}=\left[\frac{(K+1)(K+2)}{2}\right]^2 \text{ as desired}$$

f) Explain why these steps show that this formula is true whenever $n$ is a positive integer.

We have completed both the basis step and the inductive step, so by mathematical induction, the statement is true for every positive integer $n$

(2) We use the formula $1^3+2^3+\cdots n^3=\left[\frac{n(n+1)}{2}\right]^2$ for $n=100$

$1^3+2^3+\cdots+100^3=\left[\frac{100*101}{2}\right]^2=[5050]^2=\boxed{25,502,500}$

(3) Let $P(n)$ be the statement that

$$\frac{1}{1*2} + \frac{1}{2*3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$ for the positive integer $n$.

BASIS STEP. We prove that $P(1)$ is true

$P(1)$ is the statement that $\frac{1}{1*2} = \frac{1}{1+1}$

Since both sides of $P(1)$ are equal (to $\frac{1}{2}$), $P(1)$ is true.

INDUCTIVE STEP. We prove that $P(K)$ implies $P(K+1)$, where $K$ is a positive integer.

$P(K)$ is the statement that

$$\frac{1}{1*2} + \frac{1}{2*3} + \cdots + \frac{1}{K(K+1)} = \frac{K}{(K+1)}$$ (inductive hypothesis)

$P(K+1)$ is the statement that

$$\frac{1}{1*2} + \frac{1}{2*3} + \cdots + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)} = \frac{K+1}{K+2}.$$

To show that $P(K+1)$ is true we show that LHS = RHS.

$$\underbrace{\boxed{\frac{1}{1*2} + \frac{1}{2*3} + \cdots + \frac{1}{K(K+1)}} + \frac{1}{(K+1)(K+2)} = \boxed{\frac{K}{K+1}}}_{= \text{ by inductive hypothesis.}} + \frac{1}{(K+1)(K+2)} \underset{\text{algebra}}{\overset{=}{\downarrow}} \frac{K(K+2)+1}{(K+1)(K+2)} =$$

$$\frac{K^2+2K+1}{(K+1)(K+2)} = \frac{(K+1)(K+1)}{(\cancel{K+1})(K+2)} = \frac{K+1}{K+2}$$ as desired.

We have completed the basis step and the inductive step, so by mathematical induction the statement is true for every positive integer $n$.