2.4 COMPLEX NUMBERS

What You Should Learn

- Use the imaginary unit *i* to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

The Imaginary Unit *i*

The Imaginary Unit i

$x^2 + 1 = 0$ has no real solution

has complex solution

Imaginary unit i

$$i = \sqrt{-1}$$

where $i^2 = -1$

The Imaginary Unit *i*

Complex number in **standard form:**

$$\boldsymbol{a + bi}$$

Example:

Complex number $-5 + \sqrt{-9}$

Standard form: $-5 + 3i$

$$-5 + \sqrt{-9} = -5 + \sqrt{3^2(-1)} = -5 + 3\sqrt{-1} = -5 + 3i.$$

The Imaginary Unit *i*

Definition of a Complex Number

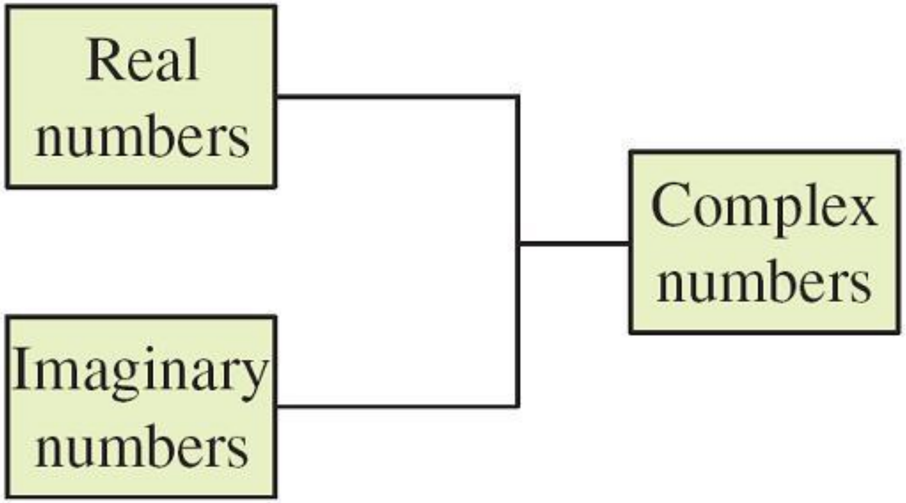
If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

$$a \qquad + \qquad bi$$

Real part   imaginary part

The Imaginary Unit *i*

$$a = a + 0i$$



The Imaginary Unit i

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

EXAMPLE:

$$x + 4i = -2 + bi$$

$$\Rightarrow \begin{cases} x = -2 \\ 4 = b \end{cases}$$

Operations with Complex Numbers

Operations with Complex Numbers

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

EXAMPLE:

$2 + 5i - (4 + 7i)$

$= (2 - 4) + (5 - 7)i$

$= -2 + (-2)i$

$= -2 - 2i$

Add/Subtract binomials

$2 + 5x - (4 + 7x)$

$= (2 - 4) + (5 - 7)x$

$= -2 + (-2)x$

$= -2 - 2x$

Operations with Complex Numbers

The **additive identity** in the complex number system is zero (the same as in the real number system).

The **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi.$$

Additive inverse

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

Example 1 – *Adding and Subtracting Complex Numbers*

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ — Remove parentheses.

$= (4 + 1) + (7i - 6i)$ — Group like terms.

$= 5 + i$ — Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ — Remove parentheses.

$= (1 - 4) + (2i - 2i)$ — Group like terms.

$= -3 + 0$ — Simplify.

$= -3$ — Write in standard form.

Example 1 – *Adding and Subtracting Complex Numbers* cont'd

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3i - 3i - 5i)$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

Operations with Complex Numbers

$(a + bi)(c + di) = a(c + di) + bi(c + di)$ Distributive Property

$= ac + (ad)i + (bc)i + (bd)i^2$ Distributive Property

$= ac + (ad)i + (bc)i + (bd)(-1)$ $i^2 = -1$

$= ac - bd + (ad)i + (bc)i$ Commutative Property

$= (ac - bd) + (ad + bc)i$ Associative Property

FOIL

Complex Numbers

Binomials

Complex Conjugates

Complex Conjugates

The product of two complex numbers can be a real number.

This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2$$

$$= a^2 - b^2(-1)$$

$$= a^2 + b^2$$

Example 3 – *Multiplying Conjugates*

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution:

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2$$

$$= 1 - (-1)$$

$$= 2$$

Example 3 – *Solution*

cont'd

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2$$

$$= 16 - 9i^2$$

$$= 16 - 9(-1)$$

$$= 25$$

Complex Conjugates

To write the quotient of *a* + *bi* and *c* + *di* in standard form, where *c* and *d* are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\left(\frac{c-di}{c-di}\right)$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}.$$

Standard form

Complex Solutions of Quadratic Equations

Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}\,i$$

The number $\sqrt{3}\,i$ is called the *principal square root* of –3.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}\,i.$$

Example 6 – *Complex Solutions of a Quadratic Equation*

Solve **(a)** $x^2 + 4 = 0$ and **(b)** $3x^2 - 2x + 5 = 0$.

Solution:

a. $x^2 + 4 = 0$ — Write original equation.

$x^2 = -4$ — Subtract 4 from each side.

$x = \pm 2i$ — Extract square roots.

b. $3x^2 - 2x + 5 = 0$ — Write original equation.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$

Quadratic Formula

Example 6 – *Solution*

cont'd

$$= \frac{2 \pm \sqrt{-56}}{6}$$ Simplify.

$$= \frac{2 \pm 2\sqrt{14}\,i}{6}$$ Write $\sqrt{-56}$ in standard form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$ Write in standard form.