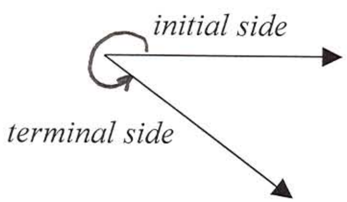
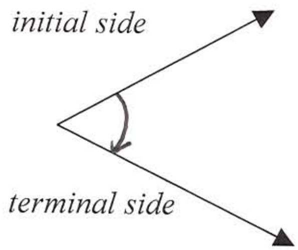
The Basics of Trigonometry

I. Definitions

A. An angle is formed by rotating a ray around its endpoint or vertex. The initial position of the ray is the initial side of the angle, while the location of the ray at the end of its rotation is the terminal side of the angle. If the rotation of an angle is counterclockwise, the angle is positive; if the rotation is clockwise, the angle is negative.

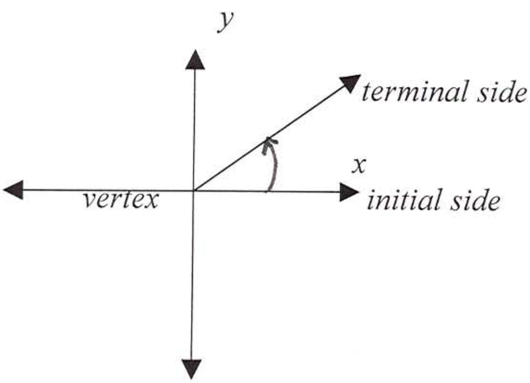


Positive angle

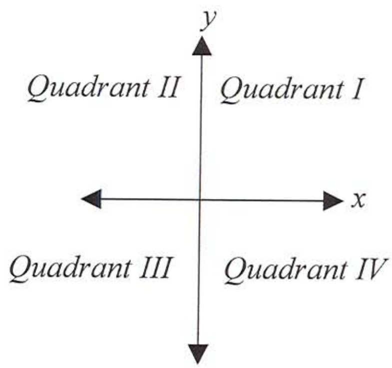


Negative angle

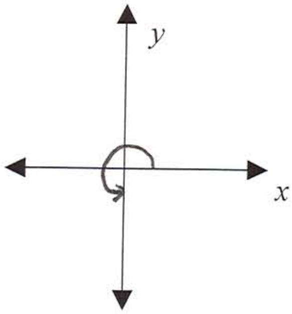
B. An angle is in standard position if its vertex is at the origin of a rectangular coordinate system and its initial side is along the positive x-axis. An angle in standard position is said to lie in the quadrant where its terminal side lies.



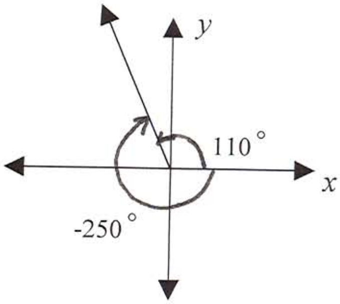
Angle in standard position



C. An angle in standard position whose terminal side coincides with the x-axis or y-axis is called a quadrantal angle. Two angles with the same initial side and the same terminal side, but different amounts of rotation, are called coterminal angles.

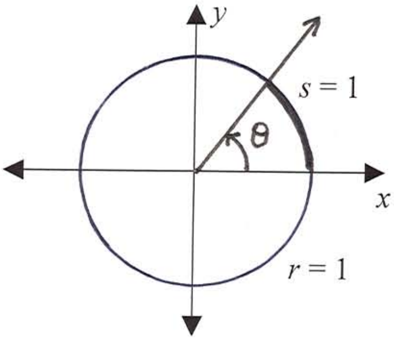


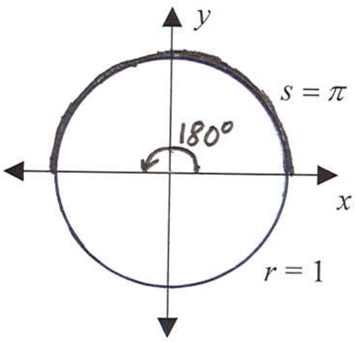
Quadrantal angle



Coterminal angles

D. Suppose a circle has radius $r > 0$. Let θ be a central angle of the circle. If θ cuts off an arc of length s on the circle, then the radian measure of θ is $\theta = \frac{s}{r}$. The radian measure of an angle is the ratio of the arc length cut by the angle to the radius of the circle. For this reason, a radian measure is a real number without any units.





Radian measure of $\theta = \frac{1}{1} = 1$

θ has radian measure 1

$180^\circ = \frac{\pi}{1} = \pi$ radians

Also, 1 radian $= \frac{180^\circ}{\pi} \approx 57.3^\circ$

Example: Convert 150° to radian measure and $\frac{7\pi}{4}$ to degree measure.

$$150^\circ = 150^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{5\pi}{6} \text{ and } \frac{7\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 7(45^\circ) = 315^\circ$$

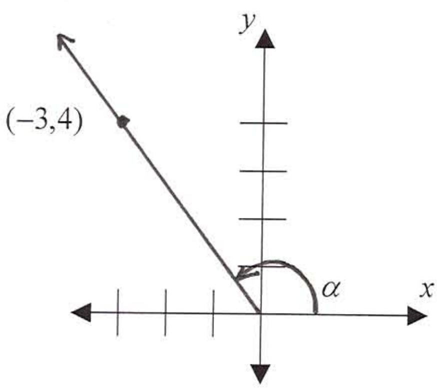
E. Let $P(x, y)$ be a point other than the origin on the terminal side of an angle θ in standard position. Let r be the distance from the origin to $P(x, y)$; ie $r = \sqrt{x^2 + y^2}$. Then the trigonometric ratios of θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Example: The terminal side of an angle α goes through the point $(-3, 4)$. Find the values of the trigonometric ratios of α.



$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

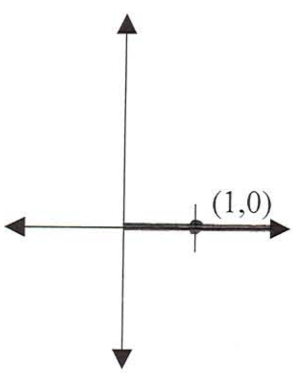
$$\sin\alpha = \frac{4}{5} \qquad \csc\alpha = \frac{5}{4}$$

$$\cos\alpha = \frac{-3}{5} \qquad \sec\alpha = \frac{5}{-3}$$

$$\tan\alpha = \frac{4}{-3} \qquad \cot\alpha = \frac{-3}{4}$$

II. Trigonometric ratios for special angles

A. 0°



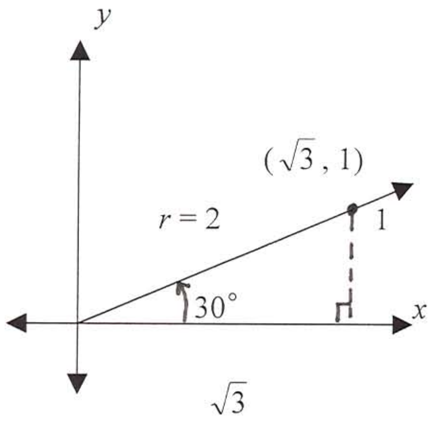
$$r = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$$

$$\sin 0^\circ = \frac{0}{1} = 0 \qquad \csc 0^\circ = \frac{1}{0} = und$$

$$\cos 0^\circ = \frac{1}{1} = 1 \qquad \sec 0^\circ = \frac{1}{1} = 1$$

$$\tan 0^\circ = \frac{0}{1} = 0 \qquad \cot 0^\circ = \frac{1}{0} = und$$

B. 30°



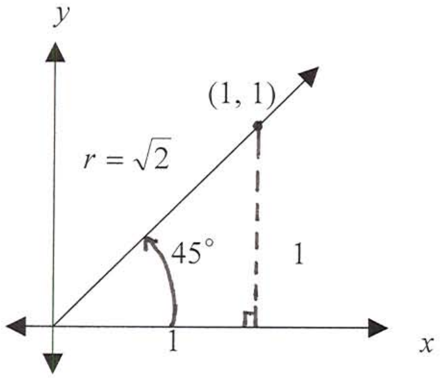
Recall from geometry that in a $30^\circ - 60^\circ - 90^\circ$ triangle, the hypotenuse is always twice as long as the shortest side (side opposite the 30° angle) and the medium side (side opposite the 60° angle) is $\sqrt{3}$ times as long as the shortest side.

$\sin 30^\circ = \frac{1}{2}$ $\qquad$ $\csc 30^\circ = 2$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$ $\qquad$ $\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\qquad$ $\cot 30^\circ = \sqrt{3}$

C. 45°



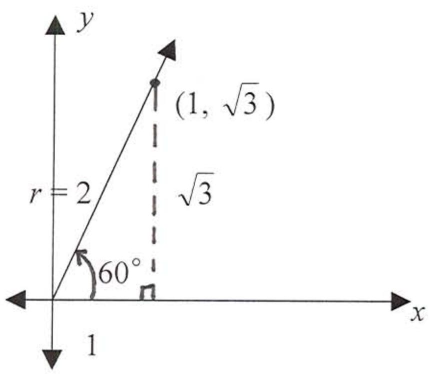
Recall from geometry that in a $45^\circ - 45^\circ - 90^\circ$ triangle, the legs are the same length and the hypotenuse is $\sqrt{2}$ times as long as either of the legs.

$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad$ $\csc 45^\circ = \sqrt{2}$

$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad$ $\sec 45^\circ = \sqrt{2}$

$\tan 45^\circ = \frac{1}{1} = 1$ $\qquad$ $\cot 45^\circ = 1$

D. 60°

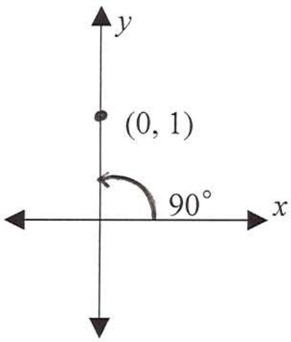


$\sin 60^\circ = \frac{\sqrt{3}}{2}$ $\qquad$ $\csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\cos 60^\circ = \frac{1}{2}$ $\qquad$ $\sec 60^\circ = 2$

$\tan 60^\circ = \sqrt{3}$ $\qquad$ $\cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

E. 90°



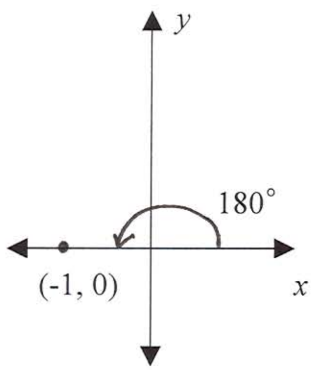
$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

$$\sin 90° = \frac{1}{1} = 1 \qquad \csc 90° = 1$$

$$\cos 90° = \frac{0}{1} = 0 \qquad \sec 90° = \frac{1}{0} = \textit{undefined}$$

$$\tan 90° = \frac{1}{0} = \textit{undefined} \qquad \cot 90° = \frac{0}{1} = 0$$

F. 180°



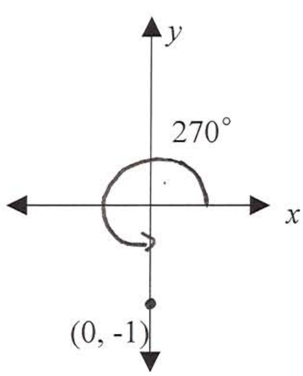
$$r = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$$

$$\sin 180° = \frac{0}{1} = 0 \qquad \csc 180° = \frac{1}{0} = \textit{undefined}$$

$$\cos 180° = \frac{-1}{1} = -1 \qquad \sec 180° = \frac{1}{-1} = -1$$

$$\tan 180° = \frac{0}{-1} = 0 \qquad \cot 180° = \frac{-1}{0} = \textit{undefined}$$

G. 270°



$$r = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$$

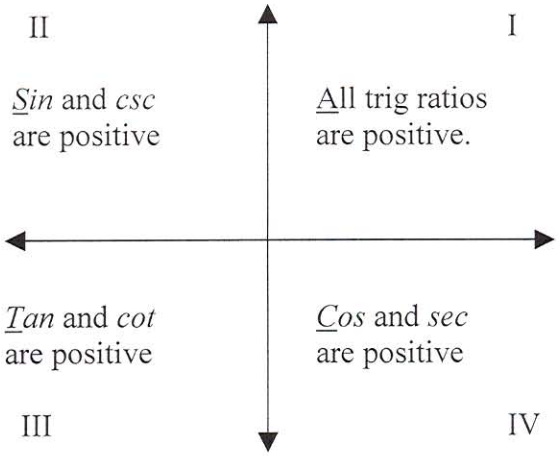
$$\sin 270° = \frac{-1}{1} = -1 \qquad \csc 270° = -1$$

$$\cos 270° = \frac{0}{1} = 0 \qquad \sec 270° = \frac{1}{0} = \textit{undefined}$$

$$\tan 270° = \frac{-1}{0} = \textit{undefined} \qquad \cot 270° = \frac{0}{-1} = 0$$

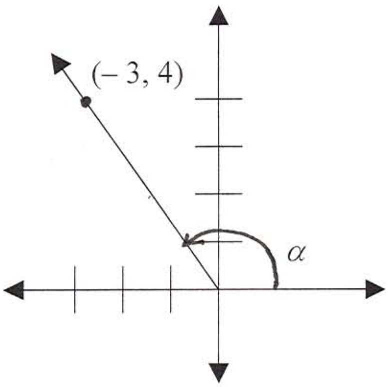
III. Trigonometric ratios using quadrants and reference angles

A. Trigonometric ratios and quadrants



Example 1: Suppose that $\cos\alpha = -\frac{3}{5}$ and α lies in quadrant II. Find the other trigonometric ratios for α.

$\cos\alpha = -\frac{3}{5} = \frac{x}{r} \Rightarrow$ since $r > 0$, $x = -3$ and $r = 5$. To find y, $r = \sqrt{x^2 + y^2} \Rightarrow$ $5 = \sqrt{(-3)^2 + y^2} \Rightarrow 25 = 9 + y^2 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$. Since α lies in quadrant II, $y > 0 \Rightarrow y = 4$.



$\sin\alpha = \frac{4}{5}$ $\csc\alpha = \frac{5}{4}$

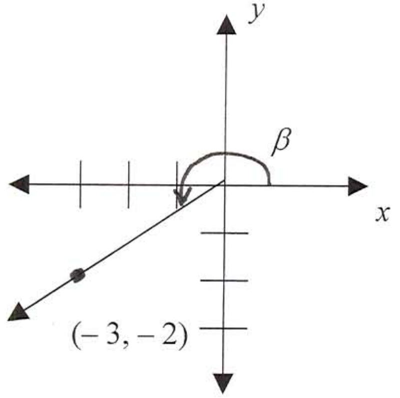
$\cos\alpha = \frac{-3}{5}$ $\sec\alpha = \frac{5}{-3}$

$\tan\alpha = \frac{4}{-3}$ $\cot\alpha = \frac{-3}{4}$

Example 2: Suppose that $\tan\beta = \dfrac{2}{3}$ and $\sin\beta < 0$. Find the other trigonometric ratios for β.

Since $\sin\beta < 0$, the $y < 0 \Rightarrow \tan\beta = \dfrac{2}{3} = \dfrac{-2}{-3} = \dfrac{y}{x} \Rightarrow x = -3$ and $y = -2$.

Thus, $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$.

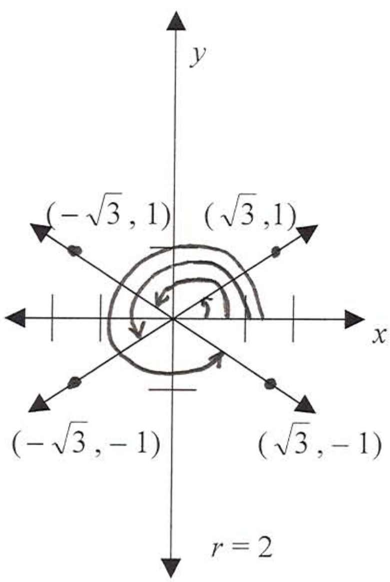


$\sin\beta = \dfrac{-2}{\sqrt{13}}$ $\qquad$ $\csc\beta = \dfrac{\sqrt{13}}{-2}$

$\cos\beta = \dfrac{-3}{\sqrt{13}}$ $\qquad$ $\sec\beta = \dfrac{\sqrt{13}}{-3}$

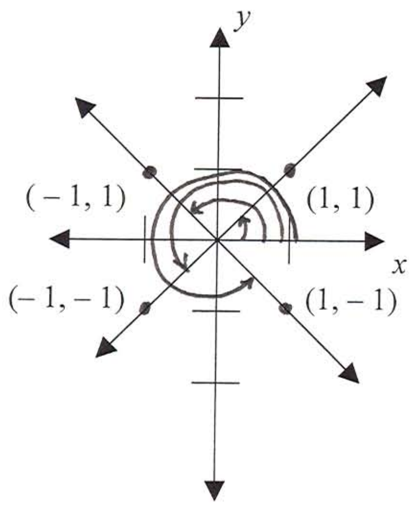
$\tan\beta = \dfrac{2}{3}$ $\qquad$ $\cot\beta = \dfrac{3}{2}$

B. 30°, 150°, 210°, 330°



	30°	150°	210°	330°
sin	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
cos	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
tan	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$
csc	2	2	-2	-2
sec	$\frac{2\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$
cot	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$

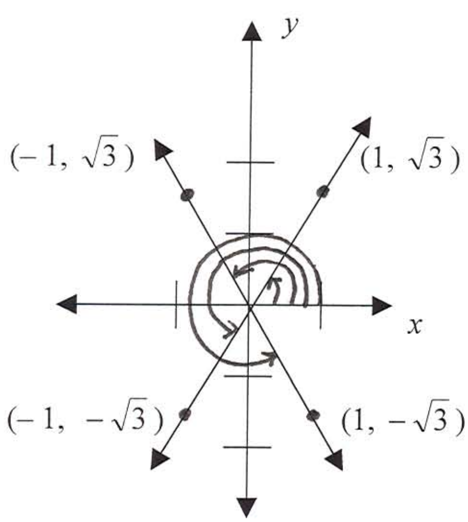
C. 45°, 135°, 225°, 315°



$r = \sqrt{2}$

	45°	135°	225°	315°
sin	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
cos	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
tan	1	– 1	1	– 1
csc	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
sec	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$
cot	1	– 1	1	– 1

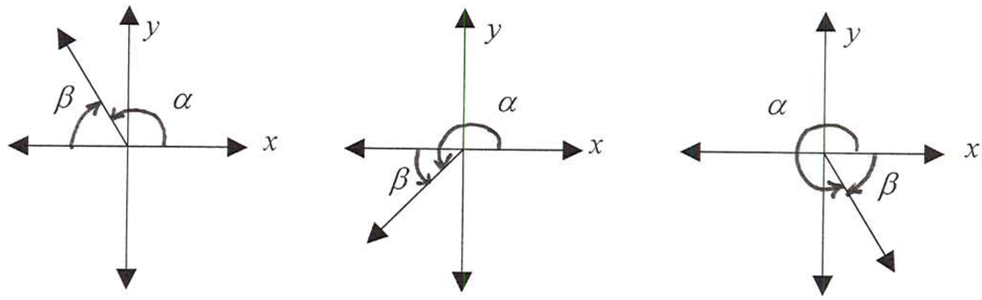
D. 60°, 120°, 240°, 300°



$r = 2$

	60°	120°	240°	300°
sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
cos	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
tan	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$
csc	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$
sec	2	– 2	– 2	2
cot	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

E. The reference angle for the angle α is the positive acute angle formed by the terminal side of α and the x-axis. The following diagrams show the reference angle β for an angle α whose terminal side lies in the second, third, or fourth quadrants. Quadrantal angles do not have a reference angle. The reference angle for an angle in the first quadrant is the given angle.



Example 1: Find the exact value of $\tan 120^\circ$.

The terminal side of the angle whose measure is 120° lies in the 2nd quadrant, so $180^\circ - 120^\circ = 60^\circ$ is the reference angle. Also, tan is negative in the 2nd quadrant. Thus, $\tan 120^\circ = -\tan 60^\circ = -\sqrt{3}$.

Example 2: Find the exact value of $\sin 210^\circ$.

The terminal side of the angle whose measure is 210° lies in the 3rd quadrant, so $210^\circ - 180^\circ = 30^\circ$ is the reference angle. Also, sin is negative in the 3rd quadrant. Thus, $\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$.

Example 3: Find the exact value of $\cos 315^\circ$.

The terminal side of the angle whose measure is 315° lies in the 4th quadrant, so $360^\circ - 315^\circ = 45^\circ$ is the reference angle. Also, cos is positive in the 4th quadrant. Thus, $\cos 315^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$.

Practice Sheet – The Basics of Trigonometry

I. Convert the following degree measures to exact radian measures:

(1) $35^\circ =$ (2) $150^\circ =$ (3) $220^\circ =$ (4) 405°

II. Convert the following radian measures to degree measures:

(1) $\frac{2}{3}\pi =$ (2) $\frac{5}{9}\pi =$ (3) $\frac{7}{18}\pi =$ (4) $\frac{8}{5}\pi =$

III. The terminal side of an angle θ goes through the point (8, – 15).

(1) $r =$ (2) $\sin\theta =$ (3) $\cos\theta =$ (4) $\tan\theta =$

IV. Suppose that $\tan\alpha = -\frac{3}{4}$ and $\cos\alpha < 0$.

(1) $\sin\alpha =$ (2) $\cos\alpha =$

V. Suppose that $\sin\beta = \frac{\sqrt{2}}{5}$ and $\tan\beta > 0$.

(1) $\cos\beta =$ (2) $\tan\beta =$

VI. Find the exact values for each of the following:

(1) $\sin 45^\circ =$ (2) $\cos 90^\circ =$ (3) $\tan 150^\circ =$ (4) $\sin 240^\circ =$

(5) $\cos 315^\circ =$ (6) $\tan 390^\circ =$ (7) $\sin 510^\circ =$ (8) $\cos(-60^\circ) =$

(9) $\tan(-135^\circ) =$ (10) $\sin(-90^\circ) =$

Solution Key for The Basics of Trigonometry

I. (1) $\frac{7}{36}\pi$ (2) $\frac{5}{6}\pi$ (3) $\frac{11}{9}\pi$ (4) $\frac{9}{4}\pi$

II. (1) 120° (2) 100° (3) 70° (4) 288°

III. (1) 17 (2) $-\frac{15}{17}$ (3) $\frac{8}{17}$ (4) $-\frac{15}{8}$

IV. (1) $\frac{3}{5}$ (2) $-\frac{4}{5}$

V. (1) $\frac{\sqrt{23}}{5}$ (2) $\frac{\sqrt{2}}{\sqrt{23}}$

VI. (1) $\frac{\sqrt{2}}{2}$ (2) 0 (3) $-\frac{\sqrt{3}}{3}$ (4) $-\frac{\sqrt{3}}{2}$

(5) $\frac{\sqrt{2}}{2}$ (6) $\frac{\sqrt{3}}{3}$ (7) $\frac{1}{2}$ (8) $\frac{1}{2}$

(9) 1 (10) -1