3.2 LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

What You Should Learn

- Recognize and evaluate logarithmic functions with base *a*.
- Graph logarithmic functions.
- Recognize, evaluate, and graph natural logarithmic functions.
- Use logarithmic functions to model and solve real-life problems.

Logarithmic Functions

Logarithmic Functions

Every function of the form $f(x) = a^x$ passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

Definition of Logarithmic Function with Base *a*

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by

$$f(x) = \log_a x$$

Read as "log base *a* of *x*."

is called the **logarithmic function with base *a*.**

Logarithmic Functions

The equations

$$y = \log_a x \quad \text{and} \quad x = a^y$$

are equivalent. The first equation is in logarithmic form and the second is in exponential form.

For example, the logarithmic equation $2 = \log_3 9$ can be rewritten in exponential form as $9 = 3^2$. The exponential equation $5^3 = 125$ can be rewritten in logarithmic form as $\log_5 125 = 3$.

Logarithmic Functions

When evaluating logarithms, remember that *a logarithm is an exponent*. This means that $\log_a x$ is the exponent to which *a* must be raised to obtain *x*.

For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

Example 1 – *Evaluating Logarithms*

Use the definition of logarithmic function to evaluate each logarithm at the indicated value of *x*.

a. $f(x) = \log_2 x$, $x = 32$

b. $f(x) = \log_3 x$, $x = 1$

c. $f(x) = \log_4 x$, $x = 2$

d. $f(x) = \log_{10} x$, $x = \frac{1}{100}$

Solution:

a. $f(32) = \log_2 32$ because $2^5 = 32$.

$= 5$

b. $f(1) = \log_3 1$ because $3^0 = 1$.

$= 0$

Example 1 – *Solution*

cont'd

c. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

Logarithmic Functions

The logarithmic function with base 10 is called the **common logarithmic function.** It is denoted by $\log_{10}$ or simply by log. On most calculators, this function is denoted by LOG.

The following properties follow directly from the definition of the logarithmic function with base *a*.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.

2. $\log_a a = 1$ because $a^1 = a$.

3. $\log_a a^x = x$ and $a^{\log_a x} = x$ — Inverse Properties

4. If $\log_a x = \log_a y$, then $x = y$. — One-to-One Property

Graphs of Logarithmic Functions

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 5 – *Graphs of Exponential and Logarithmic Functions*

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

Example 5(a) – *Solution*

For $f(x) = 2^x$, construct a table of values.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

By plotting these points and connecting them with a smooth curve, you obtain the graph shown in Figure 3.14.

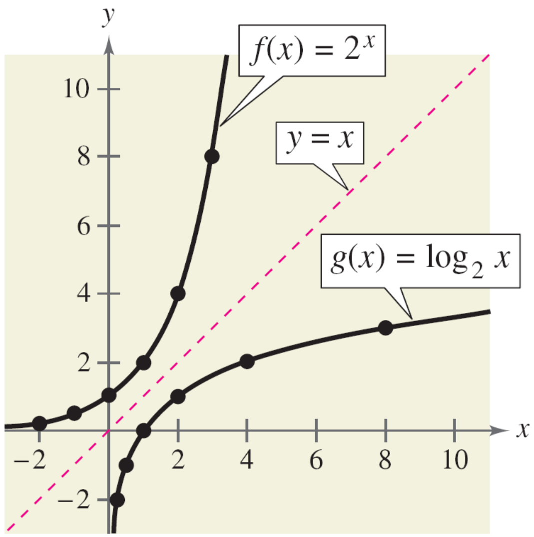


Figure 3.14

Example 5(b) – *Solution*

cont'd

Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve.

The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.14.

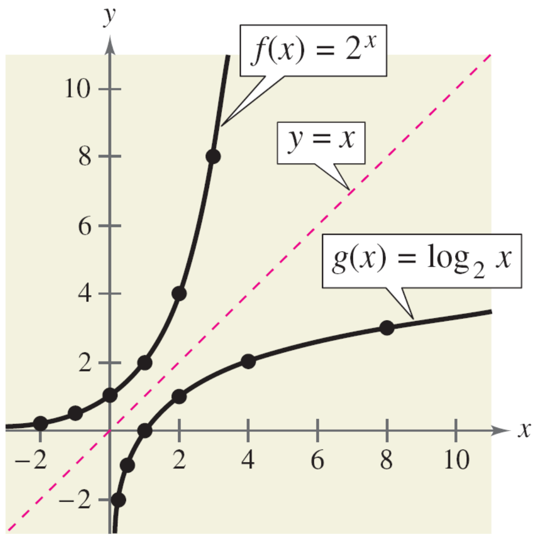


Figure 3.14

Graphs of Logarithmic Functions

The nature of the graph in Figure 3.15 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$.

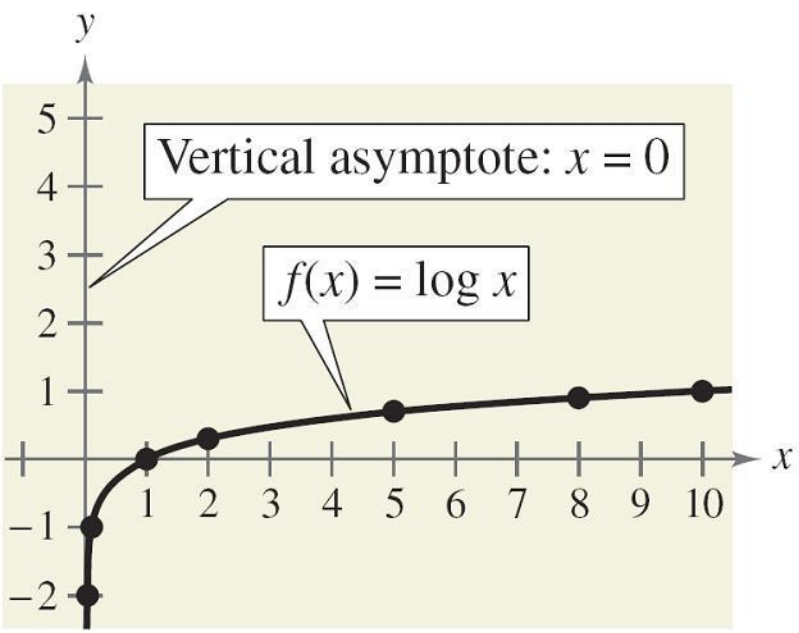


Figure 3.15

Graphs of Logarithmic Functions

The basic characteristics of logarithmic graphs are summarized in Figure 3.16.

Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- x-intercept: (1, 0)
- Increasing
- One-to-one, therefore has an inverse function

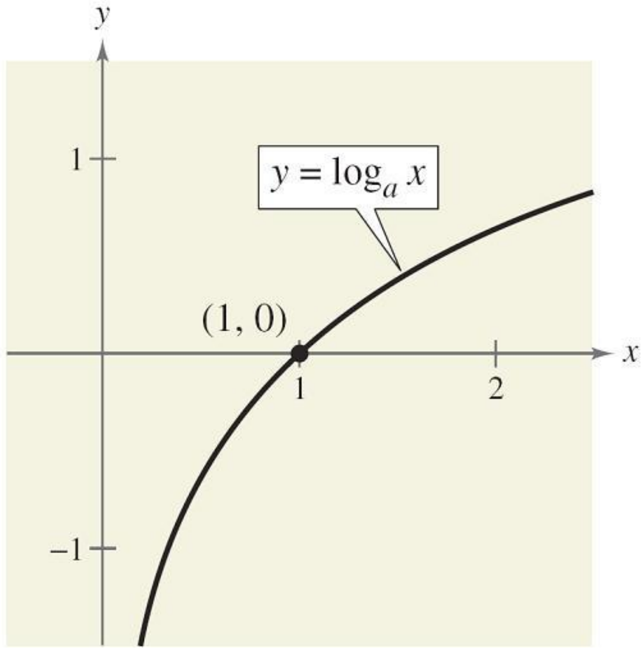


Figure 3.16

Graphs of Logarithmic Functions

- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$).
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$.

The basic characteristics of the graph of $f(x) = a^x$ are shown below to illustrate the inverse relation between $f(x) = a^x$ and $g(x) = \log_a x$.

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$).

The Natural Logarithmic Function

The Natural Logarithmic Function

We will see that $f(x) = e^x$ is one-to-one and so has an inverse function.

This inverse function is called the **natural logarithmic function** and is denoted by the special symbol ln *x*, read as "the natural log of *x*" or "el en of *x*." Note that the natural logarithm is written without a base. The base is understood to be *e*.

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \quad x > 0$$

is called the **natural logarithmic function.**

The Natural Logarithmic Function

The definition above implies that the natural logarithmic function and the natural exponential function are inverse functions of each other.

So, every logarithmic equation can be written in an equivalent exponential form, and every exponential equation can be written in logarithmic form.

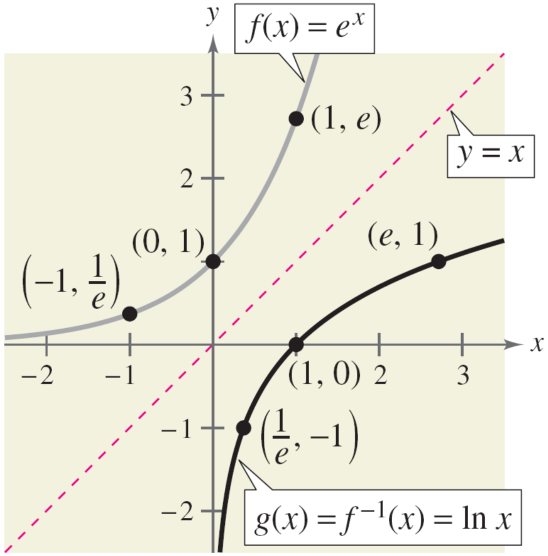
That is, $y = \ln x$ and $x = e^y$ are equivalent equations.

The Natural Logarithmic Function

Because the functions given by $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$.

This reflective property is illustrated in Figure 3.19.

On most calculators, the natural logarithm is denoted by LN, as illustrated in Example 8.



Reflection of graph of $f(x) = e^x$ about the line $y = x$.

Figure 3.19

Example 8 – *Evaluating the Natural Logarithmic Function*

Use a calculator to evaluate the function given by $f(x) = \ln x$ for each value of x.

a. $x = 2$

b. $x = 0.3$

c. $x = -1$

d. $x = 1 + \sqrt{2}$

Example 8 – *Solution*

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(2) = \ln 2$	LN 2 ENTER	0.6931472
b. $f(0.3) = \ln 0.3$	LN .3 ENTER	–1.2039728
c. $f(-1) = \ln(-1)$	LN (–) 1 ENTER	ERROR
d. $f(1+\sqrt{2}) = \ln(1+\sqrt{2})$	LN (1 + √ 2) ENTER	0.8813736

The Natural Logarithmic Function

The four properties of logarithms are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.

2. $\ln e = 1$ because $e^1 = e$.

3. $\ln e^x = x$ and $e^{\ln x} = x$ Inverse Properties

4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

Application

Example 11 – *Human Memory Model*

Students participating in a psychology experiment attended several lectures on a subject and were given an exam.

Every month for a year after the exam, the students were retested to see how much of the material they remembered.

The average scores for the group are given by the *human memory model* $f(t) = 75 - 6 \ln(t + 1)$, $0 \le t \le 12$, where t is the time in months.

Example 11 – *Human Memory Model* cont'd

a. What was the average score on the original ($t = 0$) exam?

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

Solution:

a. The original average score was

$f(0) = 75 - 6 \ln(0 + 1)$ Substitute 0 for *t*.

$= 75 - 6 \ln 1$ Simplify.

Example 11 – *Solution*

cont'd

$= 75 - 6(0)$ Property of natural logarithms

$= 75.$ Solution

b. After 2 months, the average score was

$f(2) = 75 - 6 \ln(2 + 1)$ Substitute 2 for *t*.

$= 75 - 6 \ln 3$ Simplify.

$\approx 75 - 6(1.0986)$ Use a calculator.

$\approx 68.4.$ Solution

Example 11 – *Solution*

cont'd

c. After 6 months, the average score was

$f(6) = 75 - 6\ln(6 + 1)$ Substitute 6 for *t*.

$= 75 - 6\ln 7$ Simplify.

$\approx 75 - 6(1.9459)$ Use a calculator.

$\approx 63.3.$ Solution