Graphing Exponential Functions

What is an Exponential Function?

Exponential functions are one of the most important functions in mathematics. Exponential functions have many scientific applications, such as population growth and radioactive decay. Exponential function are also used in finance, so if you have a credit card, bank account, car loan, or home loan it is important to understand exponential functions and how they work.

Exponential functions are function where the variable x is in the exponent. Some examples of exponential functions are $f(x) = 2^x$, $f(x) = 5^{x-2}$, or $f(x) = 9^{2x+1}$. In each of the three examples the variable x is in the exponent, which makes each of the examples exponential functions.

> An **Exponential Function** is a function of the form
>
> $$f(x) = b^x \quad \text{or} \quad y = b^x$$
>
> where b is called the "base" and b is a positive real number other than $1 (b > 0 \text{ and } b \neq 1)$. The domain of an exponential function is all real numbers, that is, x can be any real number.

Graphing Exponential Functions

To begin graphing exponential functions we will start with two examples. We will graph the two exponential functions by making a table of values and plotting the points. After graphing the first two examples we will take a look at the similarities and differences between the two graphs.

When creating a table of values, I always suggest starting with the numbers $x = -2, -1, 0, 1,$ and 2 because it is important to have different types of numbers, some negative, some positive, and zero.

Example 1: Graph $f(x) = 2^x$.

x	$f(x) = 2^x$
–2	$f(-2) = 2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$
–1	$f(-1) = 2^{-1} = \dfrac{1}{2^1} = \dfrac{1}{2}$
0	$f(0) = 2^0 = 1$
1	$f(1) = 2^1 = 2$
2	$f(2) = 2^2 = 4$
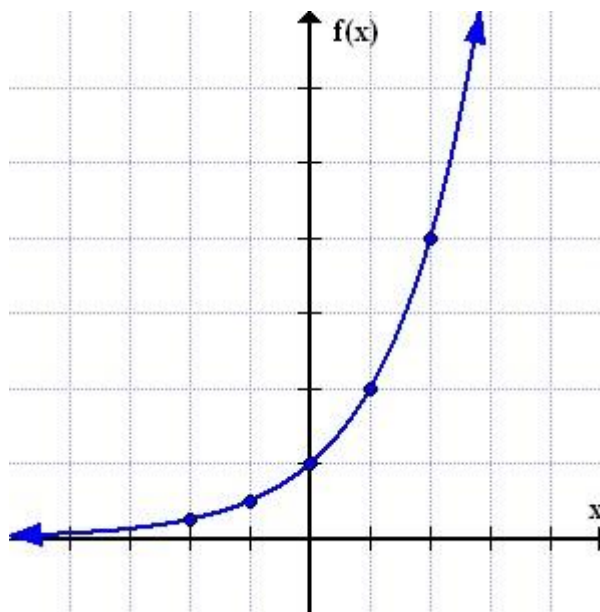
By plotting the five points in the table above and connecting the points, we get the graph shown above. Notice that as the x-values get smaller, $x = -1, -2,$ etc. the graph of the function gets closer and closer to the x-axis, but never touches the x-axis. This means that there is a horizontal asymptote at the x-axis or $y = 0$. A horizontal asymptote is a horizontal line that the graph gets closer and closer to.

Example 2: Graph $f(x) = \left(\dfrac{1}{2}\right)^x$

x	$f(x) = \left(\dfrac{1}{2}\right)^x$
−2	$f(-2) = \left(\dfrac{1}{2}\right)^{-2} = \dfrac{1^{-2}}{2^{-2}} = \dfrac{2^2}{1^2} = 4$
−1	$f(-1) = \left(\dfrac{1}{2}\right)^{-1} = \dfrac{1^{-1}}{2^{-1}} = \dfrac{2^1}{1^1} = 2$
0	$f(0) = \left(\dfrac{1}{2}\right)^0 = 1$
1	$f(1) = \left(\dfrac{1}{2}\right)^1 = \dfrac{1^1}{2^1} = \dfrac{1}{2}$
2	$f(2) = \left(\dfrac{1}{2}\right)^2 = \dfrac{1^2}{2^2} = \dfrac{1}{4}$
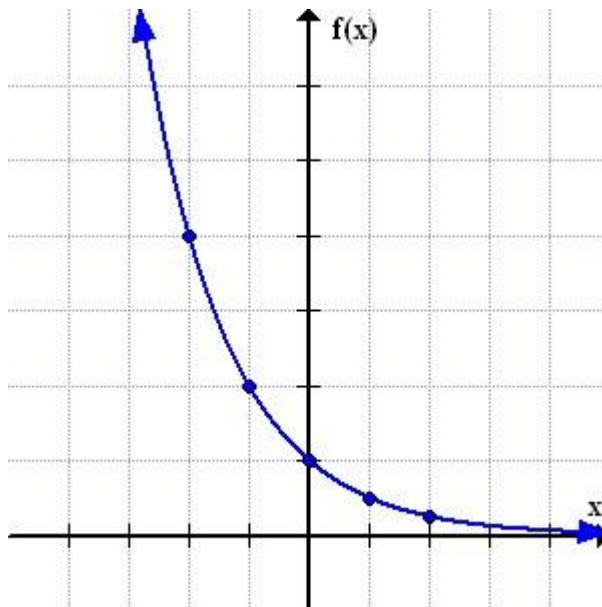
By plotting the five points in the table above and connecting the points, we get the graph shown above. Notice that as the x-values get larger, x = 1, 2, etc. the graph of the function gets closer and closer to the x-axis, but never touches the x-axis. This means that there is a horizontal asymptote at the x-axis or y = 0.

Now we can look at the similarities and differences between the graphs.

Similarities
- The domain for each example is all real numbers.
- The range for each example is all positive real numbers.
- Both graphs pass through the point (0, 1) or the y-intercept in each graph is 1.
- Both graphs get closer and closer to the x-axis, but do not touch the x-axis. So, each graph has a horizontal asymptote at the x-axis or y = 0.

Differences
- In Example 1, the graph goes upwards as it goes from left to right making it an increasing function. An exponential function that goes up from left to right is called "Exponential Growth".
- In Example 2, the graph goes downwards as it goes from left to right making it a decreasing function. An exponential function that goes down from left to right is called "Exponential Decay".

Exponential Growth or Exponential Decay

If we are given an exponential function and asked to predict if the resulting graph would be exponential growth or exponential decay, how can we correctly answer the question without actually drawing the graph? The key to correctly answering the question is to look at the base of the exponential function. Consider the following exponential functions and try to predict growth or decay by looking at the base of the function:

$$f(x) = \left(\dfrac{4}{3}\right)^x \text{ and } f(x) = \left(\dfrac{6}{5}\right)^x$$

The function with the base of 4/3 will be exponential growth and the other function with a base of 6/5 will also be exponential growth. The key to determining growth or decay depends on if the base, b, is less than one or greater than one. If the base is greater than one, b > 1, we will get growth, an increase as the graph goes from left to right. If the base is less than one, 0 < b < 1, we will get decay, a decrease as the graph goes from left to right. In the two functions used above, both 4/3 and 6/5 are greater than 1 which is why both graphs would result in exponential growth.

Summary

Here is a summary of the features of the graph of an exponential function, $f(x) = b^x$.

Features of the Graph of Exponential Functions in the Form $f(x) = b^x$ or $y = b^x$

- The domain of $f(x) = b^x$ is all real numbers.

- The range of $f(x) = b^x$ is all positive real numbers, $f(x) > 0$ or $y > 0$.

- The graph of $f(x) = b^x$ must pass through the point (0, 1) because any number, except zero, raised to the zero power is 1. The y-intercept of the graph $f(x) = b^x$ is always 1.

- The graph of $f(x) = b^x$ always has a horizontal asymptote at the x-axis ($f(x) = 0$ or $y = 0$) because the graph will get closer and closer to the x-axis but never touch the x-axis.

- If $0 < b < 1$ the graph of $f(x) = b^x$ will decrease from left to right and is called exponential decay.

- If $b > 1$ the graph of $f(x) = b^x$ will increase from left to right and is called exponential growth.


By using the features listed above we should be able to create a mental image of what the graph should look like before actually drawing the graph. Consider the following exponential functions and see if you can develop a mental image of the graph.

$$f(x) = \left(\frac{3}{2}\right)^x$$

The graph of f(x) should be exponential growth because b > 1. The graph should pass through the point (0, 1) and there should be a horizontal asymptote at the x-axis.

$$f(x) = \left(\frac{3}{4}\right)^x$$

The graph of f(x) should be exponential decay because b < 1. The graph should pass through the point (0, 1) and there should be a horizontal asymptote at the x-axis.
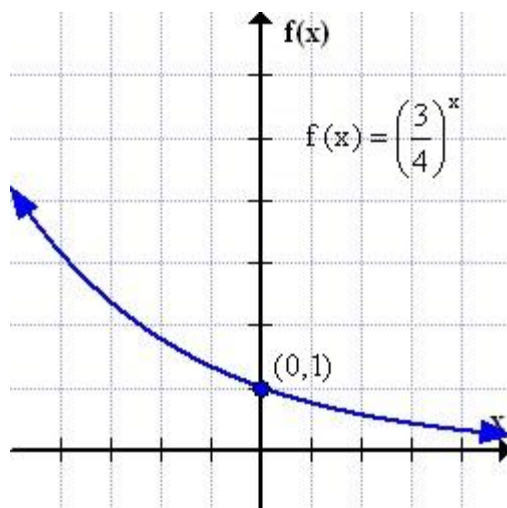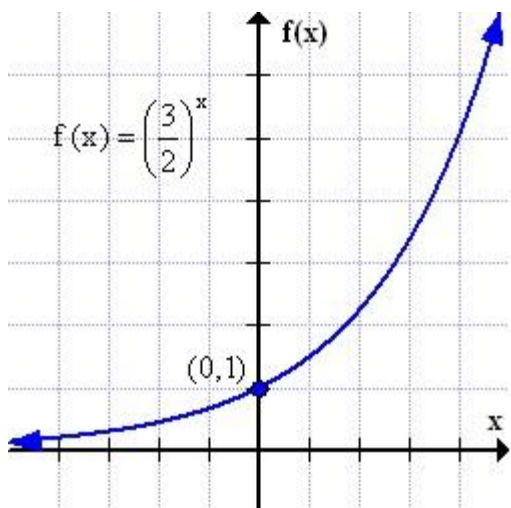
Graphing an Exponential Function with a Vertical Shift

An exponential function of the form $f(x) = b^x + k$ is an exponential function with a vertical shift. The constant k is what causes the vertical shift to occur. A vertical shift is when the graph of the function is moved up or down a fixed distance, k. When a vertical shift is applied to an exponential function, what features of the graph are affected? The features of the graph that are affected are the y-intercept, horizontal asymptote, and range. The y-intercept will move up or down a fixed amount, k, and the horizontal asymptote will also move up or down a fixed amount, k. Moving the horizontal asymptote up or down will then change the range of the function because the graph cannot touch or go below the horizontal asymptote. Consider the following examples.

Example 3 – Graph $f(x) = 2^x + 1$.

What do we know about the graph? We know that the graph is exponential growth because b > 1. The y-intercept needs to be moved up 1, meaning that the y-intercept will now be at (0, 2) instead of (0, 1). The horizontal asymptote also needs to be moved up 1, so the horizontal asymptote will be at $f(x) = 1$ or $y = 1$. Then to get a more accurate picture, we can plot some other points at x = –2, –1, 1, and 2.
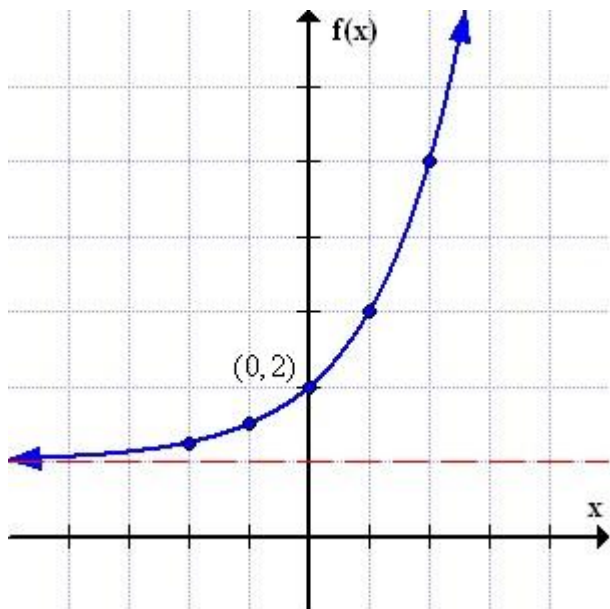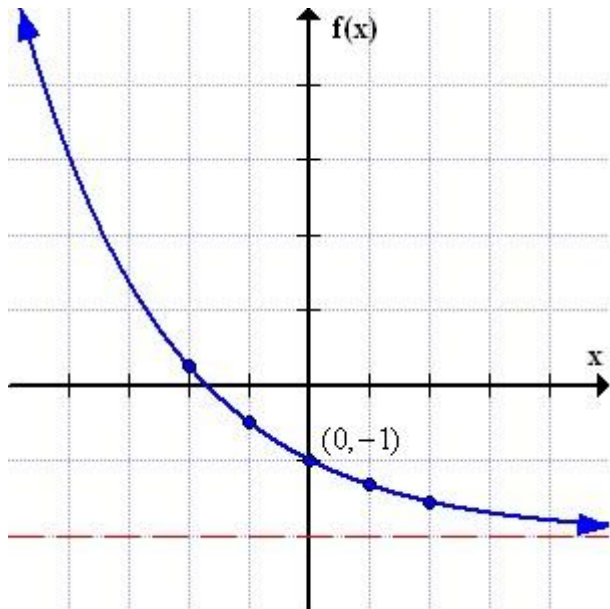
Domain: All Real Number or $(-\infty, \infty)$

Range: $f(x) > 1$, $y > 1$, or $(1, \infty)$

x	$f(x) = 2^x + 1$
–2	$f(-2) = 1.25$
–1	$f(-1) = 1.5$
1	$f(1) = 3$
2	$f(2) = 5$

Example 4: Graph $f(x) = \left(\dfrac{2}{3}\right)^x - 2$.

The graph is exponential decay because $b < 1$. The y-intercept needs to be moved down 2, meaning that the y-intercept will now be at $(0, -1)$ instead of $(0, 1)$. The horizontal asymptote also needs to be moved down 2, so the horizontal asymptote will be at $f(x) = -2$ or $y = -2$. Then to get a more accurate picture, we can plot some other points at $x = -2, -1, 1$, and 2.

Domain: All Real Number or $(-\infty, \infty)$

Range: $f(x) > -2$, $y > -2$, or $(-2, \infty)$

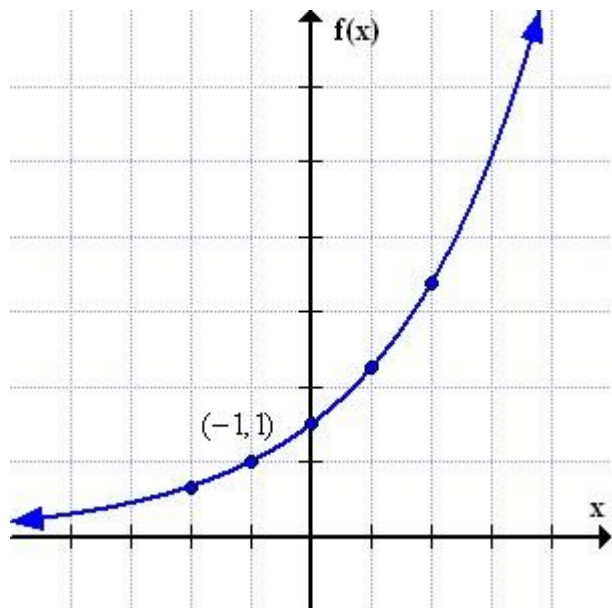| x	$f(x) = \left(\dfrac{2}{3}\right)^x - 2$
−2	$f(-2) = 0.25$
−1	$f(-1) = -0.5$
1	$f(1) = -1.\overline{33}$
2	$f(2) = -1.\overline{55}$

Graphing an Exponential Function with a Horizontal Shift

An exponential function of the form $f(x) = b^{x-h}$ is an exponential function with a horizontal shift. The constant h is what causes the horizontal shift to occur. A horizontal shift is when the graph of the function is moved to the left or right a fixed distance, h. When a horizontal shift is applied to an exponential function, what features of the graph are affected? The only feature of the graph that is affected is the y-intercept. The y-intercept will move to the left or right a fixed amount, h. This new point will NO longer be the y-intercept, but it will be a point on the graph that we can use as a reference point when visualizing the graph. The horizontal asymptote will still be at the x-axis or $y = 0$. Consider the following examples.

Note: When graphing functions with horizontal shifts, the graph will shift in the **opposite** direction of the sign used in the shift. For example, $f(x) = 3^{x+2}$ has a horizontal shift of 2 unit to the left (or backward), the opposite direction of +2. The function $f(x) = 5^{x-3}$ has a horizontal shift of 3 unit to the right (or forward), the opposite direction of −3.

Example 5: Graph $f(x) = \left(\dfrac{3}{2}\right)^{x+1}$.

What do we know about the graph? We know that the graph is exponential growth because $b > 1$. All exponential functions in the form $f(x) = b^x$ pass through the point $(0, 1)$, but in this example there is a horizontal shift, so the point $(0, 1)$ needs to shift 1 unit to the left or back 1. This means that the point $(0, 1)$ will now be $(-1, 1)$ and will no longer be the y-intercept. The horizontal asymptote will be the x-axis or $y = 0$. Then to get a more accurate picture, we can plot some other points at $x = -2, 0, 1,$ and 2.
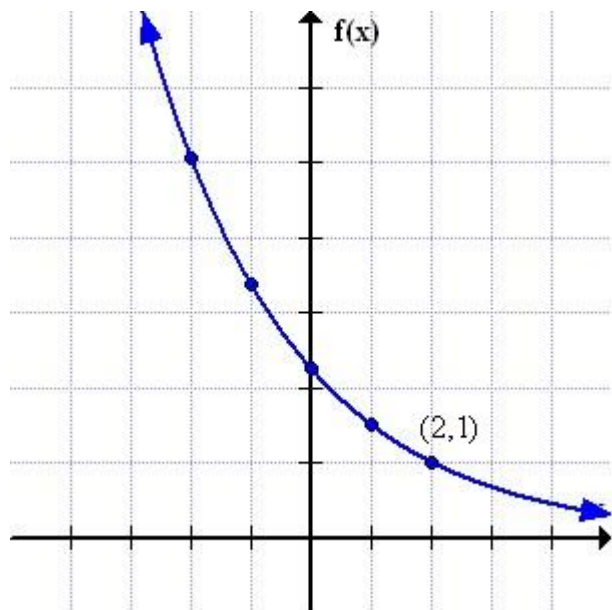

Domain: All Real Number or $(-\infty, \infty)$

Range: $f(x) > 0,\ y > 0,$ or $(0, \infty)$

x	$f(x) = \left(\dfrac{3}{2}\right)^{x+1}$
-2	$f(-2) = 0.\overline{66}$
0	$f(0) = 1.5$
1	$f(1) = 2.25$
2	$f(2) = 3.375$

Example 6: Graph $f(x) = \left(\dfrac{2}{3}\right)^{x-2}$.

The graph is exponential decay because $b < 1$. All exponential functions in the form $f(x) = b^x$ pass through the point $(0, 1)$, but in this example there is a horizontal shift, so the point $(0, 1)$ needs to shift 2 unit to the right or forward 2. This means that the point $(0, 1)$ will now be $(2, 1)$ and will no longer be the y-intercept. The horizontal asymptote will be the x-axis or $y = 0$. Then to get a more accurate picture, we can plot some other points at $x = -2, -1, 0,$ and 1.

Domain: All Real Number or $(-\infty, \infty)$

Range: $f(x) > 0,\ y > 0,$ or $(0, \infty)$

x	$f(x) = \left(\dfrac{2}{3}\right)^{x-2}$
-2	$f(-2) = 5.0625$
-1	$f(-1) = 3.375$
0	$f(0) = 2.25$
1	$f(2) = 1.5$

# Graphing Exponential Functions in the Form $f(x) = b^{x-h} + k$

Exponential functions in the form $b^{x-h} + k$ have both a vertical shift of k units and a horizontal shift of h units. When drawing graphs of exponential functions containing vertical shifts, horizontal shifts, or both, we can use the following guidelines:

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Guidelines for Graphing Exponential Functions in the Form $f(x) = b^{x-h} + k$

- Is the graph exponential growth or exponential decay? Is b > 1 or is b < 1?

- Basic exponential functions, $f(x) = b^x$, pass through the point (0, 1).

   ◇ If the graph has a vertical shift, then the point (0, 1) will move up or down k units to the point $(0, 1 \pm k)$.

   ◇ If the graph has a horizontal shift, then the point (0, 1) will move left or right h units to the point $(0 \pm h, 1)$.

   ◇ If the graph has a both a vertical shift and a horizontal shift, then the point (0, 1) will move up or down k units and left or right h units to the point $(0 \pm h, 1 \pm k)$.

- Basic exponential functions, $f(x) = b^x$, have a horizontal asymptote at the x-axis or y = 0.

   ◇ If the graph has a vertical shift, then the horizontal asymptote will move up or down k units to $y = \pm k$.

   ◇ If the graph has a horizontal shift, then the horizontal asymptote will be at the x-axis or y = 0.

   ◇ If the graph has a both a vertical shift and a horizontal shift, then the horizontal asymptote will move up or down k units to $y = \pm k$.

- To finish graphing the exponential function plot a few more point by plugging in x = –2, –1, 0, 1, and 2 as needed.

</div>

Example 7: Graph $f(x) = 2^{x-1} - 2$.

The graph is exponential growth because b > 1. This example has a vertical and horizontal shift, so the point (0, 1) needs to move 2 units down and 1 unit right to the point (1, –1). The horizontal asymptote will move down 2 units to y = –2. To finish the graph, we can plot some other points at x = –2, –1, 0, and 2.
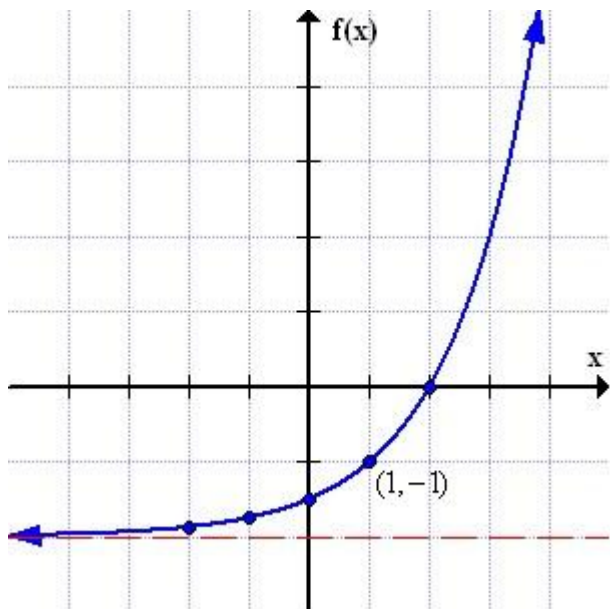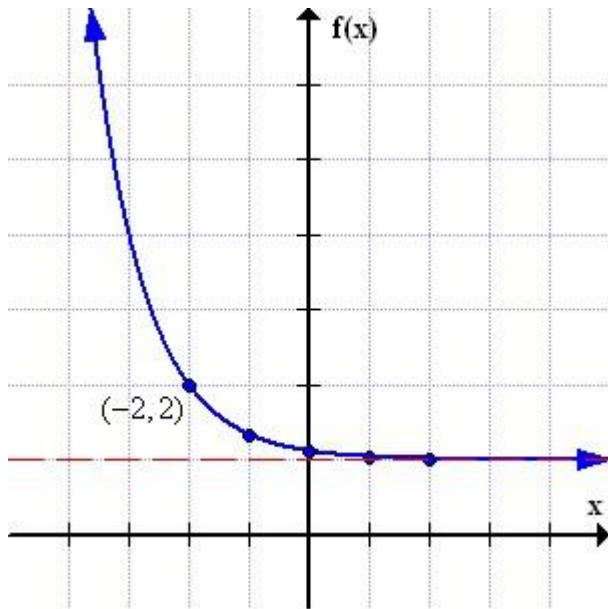

Domain: All Real Number or $(-\infty, \infty)$

Range: f(x) > –2, y > –2, or $(-2, \infty)$

x	$f(x) = 2^{x-1} - 2$
–2	$f(-2) = -1.875$
–1	$f(-1) = -1.75$
0	$f(0) = -1.5$
2	$f(2) = 0$

Example 8: Graph $f(x) = \left(\dfrac{1}{3}\right)^{x+2} + 1$.

The graph is exponential decay because $b < 1$. This example has a vertical and horizontal shift, so the point $(0, 1)$ needs to move 1 unit up and 2 units left to the point $(-2, 2)$. The horizontal asymptote will move up 1 unit to $y = 1$. To finish the graph, we can plot some other points at $x = -1, 0, 1,$ and 2.

Domain: All Real Number or $(-\infty, \infty)$

Range: $f(x) > 1$, $y > 1$, or $(1, \infty)$

x	$f(x) = 2^{x-1} - 2$
–2	$f(-2) = -1.875$
–1	$f(-1) = -1.75$
0	$f(0) = -1.5$
2	$f(2) = 0$

Addition Examples

If you would like to see more examples of graphing exponential functions, just click on the link below.

Additional Examples

Practice Problems

Now it is your turn to try a few practice problems on your own. Work on each of the problems below and then click on the link at the end to check your answers.

Problem 1: Graph $f(x) = 3^{x-1} - 2$

Problem 2: Graph $f(x) = 0.4^x - 1$

Problem 3: Graph $f(x) = \left(\dfrac{2}{3}\right)^{x+1} - 2$

Problem 4: Graph $f(x) = \left(\dfrac{7}{4}\right)^{x+1}$

Problem 5: Graph $f(x) = 1.6^{x-2} + 1$

Solutions to Practice Problems