Cramer's Rule

Cramer's rule involves using determinants of matrices to solve systems. Before we can apply this rule, we must understand how to find the determinant of a matrix. A matrix is just a rectangular arrangement of numbers. Square brackets are used around the arrangement.

$$\begin{bmatrix} 8 & -2 \\ 12 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 11 & 7 & 8 \\ 2 & 3 & 0 \\ -5 & 7 & -2 \\ 1.3 & 5 & -3 \end{bmatrix}$$

These are all examples of matrices. We identify matrices by the number of rows and columns. The rows are horizontal and the columns are vertical. The matrix on the far left is a 2x2 matrix. The matrix in the middle is a 3x1 matrix (the number of rows comes first and then the number of columns). The one on the right is a 4x3 matrix.

If the matrix has the same number of rows as it does columns it is called a square matrix. All square matrices have a determinant.

The determinant of a 2x2 matrix is denoted by $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

To evaluate a 2x2 determinant use $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

Notice that a matrix has square brackets, but a determinant has vertical lines.

Cramer's Rule

The solution of the system: $\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix}$ is given by $x = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ $\quad y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

If the determinant in the denominator is zero, then one of two things is implied:

1. If all the determinants in the above formulas are zero (both numerators and the denominator), then the system is dependent.
2. If the denominator is zero and there is at least one numerator that is not zero, then the system is inconsistent.

We will solve a system using Cramer's Rule.

ex: 6x – 5y = – 23
3x + 3y = 16

$$x = \frac{\begin{vmatrix} -23 & -5 \\ 16 & 3 \end{vmatrix}}{\begin{vmatrix} 6 & -5 \\ 3 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 6 & -23 \\ 3 & 16 \end{vmatrix}}{\begin{vmatrix} 6 & -5 \\ 3 & 3 \end{vmatrix}}$$

Notice that when looking for x, we replaced the x-term coefficients with the constants in the numerator and when looking for y, we replaced the y-term coefficients with the constants in the numerator. In the denominator, we constructed our determinant with the x and y-term coefficients. This will help us remember the rule.

$$x = \frac{(-23)(3)-(16)(-5)}{(6)(3)-(3)(-5)} = \frac{-69+80}{18+15} = \frac{11}{33} = \frac{1}{3}$$

$$y = \frac{(6)(16)-(3)(-23)}{(6)(3)-(3)(-5)} = \frac{96+69}{18+15} = \frac{165}{33} = 5$$

The denominator of each fraction is the same value (33 in this case), so you really don't need to calculate it again when looking for the second variable.

$\left(\frac{1}{3}, 5\right)$ → Answer. The check is left up to you.

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Cramer's rule can be applied to larger systems of equations, but first we need to define a 3x3 determinant.

The determinant of a 3x3 matrix is denoted by $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

To evaluate a 3x3 determinant use $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$

Each 2x2 determinant in this expansion is evaluated as discussed above. Notice the minus sign. To help remember the signs we may use the *matrix of signs*. This sign matrix tells us the signs in our expansion, just look at the positions of the numbers in the determinant. So the number "a" has a plus sign in front in the expansion, "b" has a minus sign and "c" has a plus sign.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

To get the 2x2 determinant parts, we "cross out" the row and column containing the given element of our expansion column/row. The 2x2 part left over is what we multiply by that element. For example, if we want the 2x2 determinant that goes along with *b*, we would cross out the second row and the first column, what is left over is the 2x2 determinant containing *d* & *g* in the 1st row and *f* & *i* in the 2nd row.

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

We may evaluate a determinant by expanding over any row or column. Here, we expanded over the first column. If you expand over another row or column, use the matrix of signs to get the operations (signs) in the expansion correct.

Cramer's Rule

The solution of the system $\begin{Bmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{Bmatrix}$ is given by $x = \dfrac{D_x}{D}$ $\quad y = \dfrac{D_y}{D}$ $\quad z = \dfrac{D_z}{D}$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $\quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

ex: $-x + 2y + 3z = -7$
$-4x - 5y + 6z = -13$
$7x - 8y - 9z = 39$

$$D = \begin{vmatrix} -1 & 2 & 3 \\ -4 & -5 & 6 \\ 7 & -8 & -9 \end{vmatrix} = -1\begin{vmatrix} -5 & 6 \\ -8 & -9 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 3 \\ -8 & -9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ -5 & 6 \end{vmatrix}$$

$$D = -1\left[(-5)(-9) - (6)(-8)\right] + 4\left[(2)(-9) - (3)(-8)\right] + 7\left[(2)(6) - (3)(-5)\right]$$

$$D = -1(45 + 48) + 4(-18 + 24) + 7(12 + 15) = \underline{120}$$

$$D_x = \begin{vmatrix} -7 & 2 & 3 \\ -13 & -5 & 6 \\ 39 & -8 & -9 \end{vmatrix} = -7\begin{vmatrix} -5 & 6 \\ -8 & -9 \end{vmatrix} - (-13)\begin{vmatrix} 2 & 3 \\ -8 & -9 \end{vmatrix} + 39\begin{vmatrix} 2 & 3 \\ -5 & 6 \end{vmatrix} = -7(45 + 48) + 13(-18 + 24) + 39(12 + 15) = \underline{480}$$

$$D_y = \begin{vmatrix} -1 & -7 & 3 \\ -4 & -13 & 6 \\ 7 & 39 & -9 \end{vmatrix} = -1\begin{vmatrix} -13 & 6 \\ 39 & -9 \end{vmatrix} - (-4)\begin{vmatrix} -7 & 3 \\ 39 & -9 \end{vmatrix} + 7\begin{vmatrix} -7 & 3 \\ -13 & 6 \end{vmatrix} = -1(117 - 234) + 4(63 - 117) + 7(-42 + 39) = \underline{-120}$$

$$D_z = \begin{vmatrix} -1 & 2 & -7 \\ -4 & -5 & -13 \\ 7 & -8 & 39 \end{vmatrix} = -1\begin{vmatrix} -5 & -13 \\ -8 & 39 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -7 \\ -8 & 39 \end{vmatrix} + 7\begin{vmatrix} 2 & -7 \\ -5 & -13 \end{vmatrix} = -1(-195 - 104) + 4(78 - 56) + 7(-26 - 35) = \underline{-40}$$

$x = \dfrac{480}{120} = 4$ $\qquad y = \dfrac{-120}{120} = -1$ $\qquad z = \dfrac{-40}{120} = -\dfrac{1}{3}$ $\qquad$ (4, –1, –1/3) → Our answer is an ordered triple. The check is left to you.

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