What Community College Developmental Mathematics Students Understand about Mathematics, Part 2: The Interviews

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In a prior issue of *MathAMATYC Educator*, we reported on our efforts to find out what community college developmental mathematics students understand about mathematics (Stigler, Givvin, & Thompson, 2010). Our work painted a distressing picture of students' mathematical knowledge. No matter what kind of mathematical question we asked, students tended to respond with computational procedures, which they often applied inappropriately and incorrectly. Their knowledge of mathematical concepts appeared to be fragile and weakly connected to their knowledge of procedures. But we also found some reason for hope. First, we found that when students were able to provide conceptual explanations for procedures, they often produced correct answers. Second, though students rarely used reasoning on their own to solve problems, they could reason under the right conditions.

To follow up on our previous article, we conducted one-on-one interviews with a sample of community college developmental math students. These interviews were designed to further probe students' mathematical thinking, both correct and incorrect. The interviews not only corroborate our earlier findings, but also enable us to piece together a richer picture of these students, mathematically speaking. Before we describe the methods and findings of the interviews, we will present a summary of the picture we are developing. We will also speculate on how these students came to approach math in the way they do. Though the latter was not the direct object of our study, we find such speculation helpful for understanding how these students think about mathematics, and what might be done to improve their future prospects for mathematical proficiency. We rely on other research to inform this hypothetical account of how students got where they are. After we present our hypothetical account, we will present evidence from the interviews to support it.

The Making of a Developmental Mathematics Student: A Hypothetical Account

Once upon a time, the developmental mathematics students of today were young children. Like all young children, they had, no doubt, developed some measure of mathematical competence and intuition before they entered school. Research tells us that virtually all children learn to count, develop basic concepts of quantity, and develop understandings of basic mathematical operations such as addition and division or sharing (Ginsburg, 1989; NRC, 2001). Their concepts and procedures developed in tandem. For example, they naturally learned to connect rote counting procedures to concepts such as one-to-one correspondence and cardinality (Gelman & Gallistel, 1978). Procedures such as counting were constrained by concepts, and so made sense.

All this started to change once our students entered school. There, few links were constructed between the understandings they had and the symbols and rules they were taught (Hiebert, 1984). They may have encountered teachers with narrow views of what it means to know and do mathematics. These teachers viewed mathematics as primarily about computation and applying rules (Battista, 1994), or otherwise knew that mathematics should make sense but felt that notion was implicit in the procedures they presented, and therefore never made the underlying concepts explicit. In their teaching, they emphasized procedures and paid relatively little attention to conceptual connections (Schoenfeld, 1988; Hiebert et al., 2003). In the process, our students were socialized to view mathematics as a bunch of rules, procedures, and notations, all of which needed to be remembered (Schoenfeld, 1989). At the same time, they most likely gave up on the idea that mathematics was supposed to make sense, learning that to do mathematics well simply required following the steps outlined by the teacher.

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Students who were curious, or who tried to understand why algorithms worked, were often discouraged by the teacher either overtly or inadvertently. Understanding procedures takes time, and teachers have to "cover" the curriculum. Many of these students slowly changed their view of mathematics and came to view it as mainly just steps to be remembered. Once this view started to set in, and was reinforced by rewards such as high quiz and test grades, if a teacher tried to get them to slow down and understand how an algorithm worked, the students would push back, usually by ignoring the teachers' explanations. Conceptual explanations, these students felt, were just wasting their time, time they needed for practicing and memorizing the growing number of procedures for which they were responsible and rewarded. The intuitive concepts that supported their thinking and reasoning when they were younger began to atrophy, serving no purpose in the world of school mathematics.

Although most students in U.S. schools shared these experiences, most did not end up needing to take developmental mathematics courses in community college. Some students learned on their own, or through exceptional instruction, the value of connecting rules and procedures to concepts. They discovered that things that make sense are more easily remembered, and they sought out sense-making strategies on their own. These students went on to excel in mathematics. Still others, though they experienced the conceptual atrophy we've described, were able to rely on a strong memory. On their college placement test they remembered what to do without necessarily knowing why they were doing it, and they managed to land themselves into a transfer-level math course.

The community college students that are the focus of our attention fall into neither group. Without conceptual supports and without a strong rote memory, the rules, procedures, and notations they had been taught started to degrade and get buggy over time. The process was exacerbated by an ever-increasing collection of disconnected facts to remember. With time, those facts became less accurately applied and even more disconnected from the problem solving situations in which they might have been used.

The product of this series of events is a group of students whose concepts have atrophied and whose knowledge of rules and procedures has degraded. These students lack an understanding of how important (and seemingly obvious) concepts relate (e.g., that $\frac{1}{3}$ is the same as 1 divided by 3). They also show a troubling lack of the disposition to figure things out, and very poor skills for doing so when they try. This leads them to call haphazardly upon procedures (or parts of procedures) and leaves them unbothered by inconsistencies in their solutions.

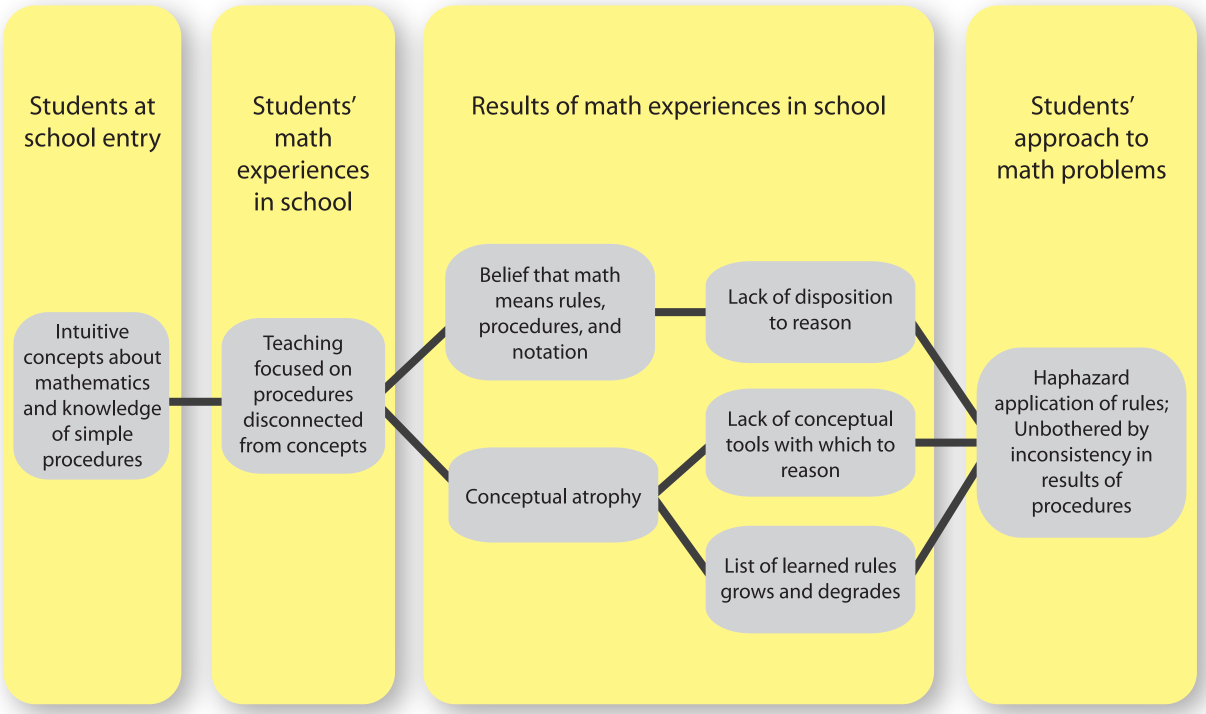

Figure 1. The making of a community college developmental math student: A hypothetical account.

Our story is summarized in Figure 1.

In the sections that follow, we describe the interviews we conducted. As we will show, these interviews are consistent with the story we have created, further enriching our view of what community college developmental mathematics students understand about mathematics. We conclude by laying out some hypotheses for how we might get community college developmental mathematics students to become successful mathematics learners, which is what we argue many have been capable of being all along.

The Interview and the Interviewees

We interviewed thirty students (15 female, 15 male) enrolled in developmental mathematics courses at a community college in the greater Los Angeles area. Students were drawn equally from Arithmetic, Pre-Algebra, and Elementary Algebra sections. (The only developmental mathematics course not included in the interviews was Intermediate Algebra.) Students ranged in age from 17 to 51 ($M = 21.6$, $SD = 6.5$). Eleven were Hispanic, nine were white, four were African-American, and six were of mixed ethnicity. For ten students, English was not the primary language spoken at home. Interviews ranged in length from 54 to 128 minutes ($M = 75.6$, $SD = 14.2$). Each student received $50 for participating.

The one-on-one interviews took place on the campus at which students were enrolled and were scheduled at students' convenience, outside class time. Guided by a protocol that was refined through extensive pilot testing (see Appendix for questions relevant to this paper), the interview set out to assess students' understanding of some key math concepts, from arithmetic through pre-algebra. We opened each interview by asking students to think about what it takes to be good at mathematics. That was followed by several mathematics problems (e.g., comparisons of and operations with decimals, comparisons of fractions and placement on a number line, solving equations with one variable, and equivalence). For each line of questioning, we anticipated possible responses and created structured follow-ups. The general pattern was to begin each line of questioning at the most abstract level and to become progressively more concrete, especially when students struggled. Each of the mathematical questions concluded with prompts that pressed for reasoning. The interviewer told students that she was more interested in their thinking about mathematical problems than she was in their answers and students were encouraged to talk through their thinking aloud as they worked. The interviewer avoided pointing out when students made mistakes, often refusing to confirm an answer as correct or incorrect when a student asked. The closing of the interview included a question about advice students would give to their teachers about how to teach math so that students would better understand it. A Livescribe Pulse pen was used to record dialog and to capture written work.

Each interview question was coded by two (or more) coders. If their coding was initially discrepant, the relevant portion of the interview was reexamined and discussed until consensus was reached.

Students' Approach to Problems: Math as a Collection of Procedures to Be Applied

When we asked what it means to be good at mathematics, 77% of students spoke to the perceived procedural nature of the subject. (Only 13% spoke of being good at math at understanding concepts.) As one prealgebra student put it, "Math is just all these steps." Other students responded in a way that not only supported the role of rules and procedures, but also discounted the role of conceptual understanding. For instance, one Elementary Algebra student stated that, "In math, sometimes you have to just accept that that's the way it is and there's no reason behind it," and another Elementary Algebra student responded that, "I don't think [being good at math] has anything to do with reasoning. It's all memorization." When we asked students to give advice to a math teacher with respect to what might be done to better promote learning, the dominant themes were about presenting material more slowly and with more repetition, and breaking down procedures into smaller steps, all of which reflect an acceptance that math is about procedures.

Consistent with those beliefs, when our interview questions asked students to solve problems, students would quickly choose a procedure they remembered from school, and then set about applying it to the situation. (This was the case even when we deliberately asked questions for which executing a procedure was not necessary.) By itself, that approach might not have been problematic. The problem lies in the fact that the procedures called upon were often either inappropriate for the situation or were executed with critical errors—errors that would surely have been caught had students understood the concept underlying the procedure or noticed that the magnitude of the answer was not reasonable. Without a conceptual understanding of the procedures in their toolbox, students were left to rely solely on a memory of which to use and how to use it. It appeared that over the years and with an increasing collection of procedures from which to draw, that memory had eroded.

This behavior of drawing on a collection of procedures is evident in students' responses to an interview item that asked them to place multiple fractions (i.e., $\frac{4}{5}$, $\frac{5}{8}$, $-\frac{3}{4}$, $\frac{5}{4}$) on a number line. Thirty percent of students set about dividing, having some recollection that it might help them with the task at hand. Although it is true that one reasonable approach to this task would be to divide numerators by denominators, convert to decimals, and use decimals to place each fraction on a number line, this is not what was done by students who chose to use division. All of the students who used division did so to create a new fraction. (The advantage of doing so was unclear; if they could place the new fraction, then why not the original?) More alarmingly, two-thirds of those who used division created a new fraction that wasn't equivalent to the original. Specifically, those students divided the denominator of a fraction by the numerator. As an illustration, Student #17 (enrolled in Prealgebra) converted $\frac{4}{5}$ to $1\frac{1}{4}$, which he then converted to $\frac{5}{4}$ without noticing (or without being bothered) that $\frac{5}{4}$ was different from the $\frac{4}{5}$ with which he started. He did the same in converting $\frac{5}{8}$ to $1\frac{3}{5}$ and then to $\frac{8}{5}$. It appeared that students' primary goal was not to write a fraction in an equivalent form, but rather to perform an operation. With the procedure complete, they accepted the resulting value as correct, with no apparent regard for whether it made any sense.

One way to interpret why students divided the denominator by the numerator is that they "forgot" the order. That hypothesis would suggest that instructors should "remind" students. Another interpretation, one that is based on a more conceptual view of mathematics, is that students aren't appropriately connecting fractions and division. If students thought about $\frac{1}{3}$ as one whole that has been divided into three parts, then how could they "forget" what to divide by what? The other concept that demands attention is that of equivalence. For a student to be able to assess whether a result in this case is correct, s/he must understand that using division to rewrite a fraction as a decimal preserves the value of the number. Perhaps Student #17 knows that $\frac{5}{4}$ isn't equivalent to $\frac{4}{5}$ but doesn't know that an appropriate result has to be equivalent.

Another problem provides further evidence of students' reliance on procedures. Students were asked to select the larger value, given $\frac{a}{5}$ and $\frac{a}{8}$. With a basic understanding of fractions (or of division), a student could answer this question with no procedure at all, yet most students chose to apply one. When the problem was posed to Student #24 (enrolled in Arithmetic), she selected $\frac{a}{8}$ as the larger of the two fractions, explaining that 8 is larger than 5. When prompted to substitute a value for *a* as a way to help her make a comparison, she substituted a 1 in place of each *a*, and then multiplied the resulting two fractions, obtaining $\frac{1}{40}$. When the interviewer asked if that process helped her to compare the two fractions, she responded, "In a way." It seemed almost as if she multiplied the two fractions *because she knew how to*, regardless of whether it could help her answer the question she had been asked.

If an effective approach to applying procedures is "ready, aim, fire," then it's as if those students fail to take aim. Students' tendency to apply procedures without thinking of the concepts that underlie them was by no means limited to fractions. A question on decimal subtraction revealed the same inclination. Students were given the problem 0.572 – 0.86, written horizontally. One student (enrolled in Elementary Algebra) wrote the problem vertically and then simply treated each column as a separate problem, subtracting the smaller value from the larger no matter its placement. For her, the procedure of subtracting smaller values from larger values applied to individual digits rather than to each value as a whole. Another student wrote the problem vertically with 0.572 above 0.86 (an approach we saw from 80% of students), and when calculating, added "1" to 0.572 so that there would be a value from which to borrow. He appeared untroubled by the change. When their problem set-up didn't initially "work," those students might have taken it as sign to reconsider the set-up itself. Had they stepped back and thought about the underlying concepts, they might have selected an appropriate procedure. Instead, students clung to a desire to make their chosen procedures work, even when the adjustments they made were not mathematically valid.

Students' approach to the subtraction problems appeared to be very mechanical. We might assume that had they been given the problem 5 – 8, at least some would have correctly responded –3, or at least recognized that something about the order of the numbers must be addressed. However, students failed to connect the problem they had been asked to set up to simpler problems (such as 5 – 8) that might have helped them decide on a sensible course of action. This might be because they don't have the habit of investigating possible procedures using simpler problems, or maybe they don't think the rules for operating with integers will necessarily apply when operating with decimals.

To further investigate the degree to which students clung to familiar procedures (even when they were the most cumbersome option) we presented students with a series of multiplication problems and asked that they do the calculations mentally. They were as follows:

10 × 3 =

10 × 13 =

20 × 13 =

30 × 13 =

31 × 13 =

29 × 13 =

22 × 13 =

The series was designed to see if students would make use of decomposition and the distributive property, or perhaps rely on answers to earlier problems in the series to help them solve later ones. Either method would have made it easier to perform the calculation mentally by reducing the load on working memory. Seventy-three percent of students never used decomposition and the distributive property; 77% never relied on answers to earlier problems in the series. (Sixty-three percent never used either of the two approaches.) The standard algorithm was the most frequent approach chosen, with 80% of students using it at least once and 20% of students beginning to use it as early as 10 × 3. Some students enacted the algorithm with fingers in the air (or on the desk) and a few even "erased" when necessary, thus demonstrating their reliance not only on the algorithm itself but also on the method of carrying it out.

In many cases, answers to adjacent mental multiplication problems could have prompted students to question their work, but it very often did not. Of the 23 students who made at least one error in the series of questions, 74% made an error that was sufficiently inconsistent with their other answers that had they compared their solutions, the mistake should have been caught. Students #5 and #10 (both enrolled in Elementary Algebra) are examples. The fact that even the most egregiously incorrect answers did not alarm students reinforces for us the commitment of students to procedures they have been taught, even if the results make no sense.

Student #5

10 × 3 = 30

10 × 13 = ~~113~~ 130

20 × 13 = 86

30 × 13 = 120

31 × 13 = 123

29 × 13 = 116

22 × 13 = 92

Student #10

10 × 3 = 30

10 × 13 = 130

20 × 13 = 260

22 × 13 = 52

30 × 13 = 120

31 × 13 = 124

29 × 13 = 126

It's unclear whether students considered that there might be an easier way to solve the series of mental multiplication problems. But it appeared as if students relied not just on procedures, but on a single, familiar procedure. Perhaps they thought that if some people find such problems easy or can calculate them quickly, it's because they can simply keep track of partial products better, and not that they use a more efficient strategy. We would argue that if instruction includes an exploration of multiple solution methods and analysis of how they're related, students might come to think of procedures more flexibly. (In a later section we'll include an example of what can happen when this is done.)

It appeared in the interviews that procedures were sometimes memorized without any meaning at all, making it difficult for students to know when to use them across situations. We asked two direct questions of students aimed at assessing whether they knew why they were doing what they were doing. The first concerned the standard algorithm for multiplication and why we put a "0" (or a "*" or a blank) in the rightmost position of the second partial product, as illustrated in figure 2.

```
  22
× 13
  66
 220
 286
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Figure 2. Example of the standard algorithm for multiplication, with a zero in the rightmost position in the second partial product

Following are examples of student responses:

I guess I'm just used to it. My teacher always says to write a zero….You know, I don't really know the answer to why we can't, but I'm already programmed to do it like that, so. I wish I knew. I know there's a proper term for it, but I don't really know the term. But you can't [right align it]. Because you just get a different answer... You get the wrong answer. *(Student #4, enrolled in Prealgebra)*

It's just being taught, you know? Each time you go down for the next number you put a zero. So if there's a third number, I'd put two zeros and you keep going… [If you don't put the 0] you get the wrong answer. But I don't know why. It's just something from, you know, I guess from 4th grade. They just teach you and go with it. *(Student #8, enrolled in Elementary Algebra)*

Um, it might be correct, it might not. To me it's correct because that's how I, you know, that's how I got used to it. So, I don't know how other people might view it. *(Student #24, enrolled in Arithmetic)*

I really don't know. I don't know why it's done like that but that's the way I was taught to do it and I always just did it like that. I don't know the answer to that, though. *(Student #27, enrolled in Elementary Algebra)*

The second direct question we asked was about why we align the decimal points when we find the difference between two values. The following are example responses:

I don't know. You kinda just learn to do it with the decimals. I guess you're just programmed when you learn something. I mean, I don't really fully remember decimals. But I know you have to know the decimals, unless it's multiplication or something. But I don't remember. But I guess you're just programmed. *(Student #4, enrolled in Prealgebra)*

I guess it just looks more organized and it looks easier to approach and I think I somewhere along the line I think I was just instructed to keep the decimals lined up. I don't remember where I heard that but I just, my logic just says keep them lined up to each other. *(Student #11, enrolled in Elementary Algebra)*

I think there is a reason for it, but I just can't recall right now. But I think I was supposed to line up the decimals. I mean, but then again, I got 2 [different] answers, so I don't know. I'm not sure. [Experimenter: So maybe you don't line up the decimals?] Maybe I don't. *(Student #17, enrolled in Prealgebra)*

To the first question, fewer than half of students referred to place value in their response and to the second question only a third did. We might have hoped that a student would respond that when we multiply 22 × 13, a 0 is inserted in the second partial product because that product represents the multiplication of 22 by 10, and not 22 by 1. At best, students stated the term "place value" or "placeholder" without being able to explain its relevance to the question. Just because students can recite the place value names does not mean they attach a "powers of ten" meaning to the places. The persistent neglect of magnitude of numbers leads us to conclude that students either do not understand the magnitude of numbers or that they do not use this knowledge to reason about values when they're unsure of how to proceed.

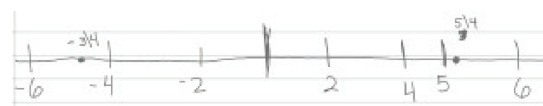
The collection of student errors we've described does more than illustrate the confused algorithmic thinking in which many students engage. It demonstrates the role that procedures play in the minds of students when they are presented math problems to solve. Given a problem, students think of a memorized procedure that might be applied to it. They don't think through the appropriateness of the procedure to the situation but proceed with a mechanical application of it. When they arrive at an answer, they're done. They don't evaluate the result's appropriateness, nor do they find reason to reconsider the procedure they chose. A desire to reconsider might have been fruitless anyway. Their lack of understanding of the meaning underlying procedures would leave them little clue as to what might be an fitting alternative.

Conceptual Atrophy and the Failure to Reason

"Conceptual atrophy" is a phrase we coined in our prior article (Stigler et al., 2010) and we use it to refer to what happens to developmental mathematics students as a result of their many years' experience of school mathematics. When students enter school, they bring with them intuitive ideas about quantity. Those intuitive ideas are often incorporated into mathematics lessons in lower elementary grades, and in later grades they frequently find their way into the introduction to a new topic. However, when it comes time to become proficient at a procedure, that conceptual basis falls away. The math instruction students then encounter frequently fails to capitalize upon students' intuitive ideas and instead emphasizes steps disconnected from meaning. Even efforts to capitalize on students' intuitions (as with estimating) often quickly turn to rules and procedures (as in "rounding to the nearest"). The result is that the potential for students to develop a stronger, more powerful conceptual grasp—a strong muscle, if you will—goes unrealized. Whatever sense of number and willingness to reason that students once had withers, and the conceptual basis that would keep procedures under control goes undeveloped. In the section above we provided examples of developmental math students' heavy reliance on procedures. Now we ask the question, do any of students' intuitive ideas about quantity remain and, if so, what are their limits?

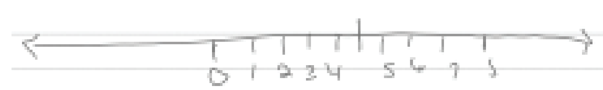
Not surprisingly, we saw a continuum in students' sense of number and ability to reason. For some students, basic concepts of number appeared lost. One would expect, for instance, that an upper-elementary student would be able to state that a proper fraction—say $\frac{1}{2}$ or $\frac{1}{3}$—is greater than 0, but less than one. Those young students could likely draw a number line containing 0 and 1, and then partition that interval into the appropriate number of segments of equal length based on the denominator. We found students, though, who struggled to place common fractions correctly on a number line. Student #21 (enrolled in Arithmetic) produced the following diagram when asked to place the numbers $-\frac{3}{4}$ and $\frac{5}{4}$ on the number line.

Student #21



Student #10 (enrolled in Elementary Algebra) started her drawing like this, and then the following discussion ensued.

Student #10



Interviewer: Can you put $\frac{4}{5}$ and $\frac{5}{8}$ on a number line?

Student: Maybe. Oh my goodness! I don't know if this is right. The number line is not my friend. Oh my goodness! I don't think this is right. Gosh! They've shown this a million times but it never processes.

I: Well, tell me what you're thinking about.

S: I don't know. I'm thinking of going to 4 but then it's just like, then between 5 but I don't know if that's right. I was thinking it was just here but I feel like that's $4\frac{1}{2}$ and not $\frac{4}{5}$. But I don't know. That's the only way that makes sense to me.

Find the derivative of $y = 2\cos(3x - \pi)$ with respect to x.

$$\frac{dy}{dx} = \boxed{6 \cdot \sin 3x}$$

"SORRY, THAT'S NOT CORRECT."

"THAT'S CORRECT."

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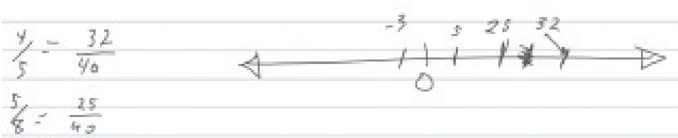
I: Okay.
S: I don't see how you can go to 4 and then somehow go to 5. Do you know what I'm saying? And then like back-tracking there to 5.
I: Okay.
S: I just don't see the connection.
I: Okay.
S: It's one of those things that I just don't understand. My teacher would show me on the board and I still won't get it.
I: The number line?
S: Just like yeah how you can– Well just the fraction or the number line. To me it's like you either go to 4 or you either go to 5, you know? You don't go [to 5] and then like you back-track [to 4], like you know? Like they should each be– like you graph 4 then you graph 5. Not you go to 4 and you go over this much. It just doesn't process to me.
I: Okay.
S: And then do you want me to $\frac{5}{8}$?
I: Where would you think that might go?
S: Honestly, I don't know. $\frac{5}{8}$. I was thinking of putting it at– I don't know why, just like putting it between at like 6 or 7, only because it's between 5 and 8.

When Student #16 (enrolled in Arithmetic) was asked to compare $\frac{4}{5}$ and $\frac{5}{8}$, he converted them to $\frac{32}{40}$ and $\frac{25}{40}$, respectively. When then asked to place the four original fractions (i.e., $\frac{4}{5}$, $\frac{5}{8}$, $-\frac{3}{4}$, and $\frac{5}{4}$) on a number line, he placed only the numerators (see the following diagram). He did not seem concerned by how the conversion affected the values he placed or by the fact that he converted only two of the four values. More importantly, his understanding of fractions (like that of Students #21 and #10, above) didn't take into account the relationship between the numerator and denominator.

Student #16



Just as basic concepts of number appeared to have atrophied, so too did understandings of basic operations. It would be reasonable to expect that any young student would be able to say that if you add two numbers to get a third, subtracting either of the initial two from the sum would leave the other. This is often shown in the "fact families" that students study in early elementary grades. This understanding is important to establishing the uniqueness of the sum of two numbers and the inverse relationship of addition and subtraction. To assess whether that understanding had withstood their years of experience in math classes, interviewees were asked to check 462 + 253 = 715 using subtraction. They nearly always subtracted the second addend from the sum (i.e., 715 – 253 = 462). However, when asked if they could have subtracted the other addend instead, some didn't know, others were skeptical, and a few claimed it would be incorrect to do so.

Student #26 (enrolled in Prealgebra) was among those who believed it is possible to subtract only one of the two addends to check the addition. The following conversation ensued after she subtracted 462 from 715.

I: How do you know to subtract the 462?
S: [LONG PAUSE] I have no idea. [LAUGHS] Because it's the biggest number?
I: Okay. Could you have subtracted the 253, or can you not do that because it's smaller?
S: You could but [LONG PAUSE] I don't think you can. I think you have to subtract the top number.
I: Okay. Why is that, that you have to subtract the top number?
S: [PAUSE] I have no idea.

Student #22 (enrolled in Arithmetic) was a skeptic. She subtracted 253 from 715. The conversation after that point was as follows:

I: How did you know to subtract 253?
S: Because it's the smaller number. Well, I don't know. My teachers said to always subtract the bottom number.
I: Okay. Could you have subtracted the top number from 715?
S: No. Well, I mean I guess you can. I don't think so, though. I don't know. 'Cus all my teachers taught me that way, so I don't think so.
I: Okay. What if you try it and see what happens?
S: So 715 – 462 = 253. [STUDENT WORKS PROBLEM ON PAPER] Oh, you can.
I: So what did you notice?
S: You could subtract any one and you'll still get either one as the answer.

Student #9 (enrolled in Elementary Algebra) had an epiphany during the interview. In the following excerpt he shares his excitement at discovering that either addend can be subtracted from the sum to obtain the other addend. It's worth noting that it took little prompting to set the student on the course of this discovery.

I: How did you know to choose 253 to subtract from 715?

S: I had a feeling you were going to ask me that right when I pulled that in for some odd reason, and I know I'm not going to give you an answer why. I don't remember why, but I just know that with some odd reason when checking– I don't know if it's always true that you pick the bottom one, 'cus I was thinking like why did I pick the bottom? How come I didn't pick the top one? I mean, what happens if you pick the top one?

I: Well, you're welcome to see, if you want.

S: Let me see that. Hold on. I never thought this until now. Wow, this is very interesting. I never thought about that. Let me see. So you're going to get 3 that becomes a 6 and 6 or 5 will get you 11 32. Oh wow, so it doesn't even matter. Is that true? Does it even matter which one you pick? 'Cus 253 is in the original equation so I guess you're right. I guess it doesn't matter which one you pick. I don't remember that. I just always remember that picking the last one for some odd reason, which is really interesting now that I learned this right now. I guess all you have to do is get your solution and then subtract it from any of these numbers. Well, I don't want to say "any" because you could be left with a negative. I want to say you could, but I'm not too sure. But I feel that I'm safe going by the answer subtracted from the bottom number and your answer should match that one to make sure that it's correct.

I: Okay.

S: So yeah.

I: Great.

S: I'm going to ask my math teacher today. It's interesting I just learned that.

We might speculate that Student #9, and others like him, were encouraged by their K–12 teachers to check their work by using inverse operations, and later in algebra class students utilized inverse relationships to solve equations. In both cases, addition and subtraction are inextricably related. However, somewhere along the line, the act became just one more procedure disconnected from its meaning. The concept students likely understood at one time had deteriorated, and was never connected to new topics.

Interestingly, we did find some instances in which students could rely on their intuitive ideas about quantity. However, when the same concepts were transferred into mathematical notation—when the math looked more like a math class—students set their conceptual understandings aside. For example, when we asked students, "What would happen if you had a number and added $\frac{1}{3}$ to it? Would it be more than what you started with, less than what you started with, the same as what you started with, or can you not tell?" Eighty-seven percent of students answered correctly that the resulting number would be larger than the original. We followed up immediately with the question, "If $a+\frac{1}{3}=x$, is x more than a, less than a, the same as a, or can you not tell?" Now students weren't so sure. Thirty percent thought the second question was unanswerable unless a and/or x was provided, in spite of the fact that 78% of those students had just correctly answered the same question, albeit without mathematical notation.

We then repeated the same two questions, but replaced addition with multiplication. That is, "What would happen if you had a number and multiplied it by $\frac{1}{3}$?" and, "If $a\times\frac{1}{3}=x$, is x more than a, less than a, the same as a, or can you not tell?" A student (enrolled in Arithmetic) who was able to agree that $\frac{1}{3}$ times a number would result in a number less than the original number used the idea of dividing by 3, a rare occurrence in this sample. However, he took a very different approach when asked the same question using the equation $a\times\frac{1}{3}=x$. He proceeded to choose a number for a, and then to multiply the number by 1 and by 3 to find x. If $a = 2$, for example, then $2 \times 1 = 2$, and $2 \times 3 = 6$. So, you end up with $x=\frac{2}{6}$, which can be simplified to $\frac{1}{3}$. He went on to show that this method works for every number, always resulting in $\frac{1}{3}$. Whatever a is, x will always be equal to $\frac{1}{3}$! This particular student commented that he has been "taught by like seven million teachers how to do this."

While it is true that the student incorrectly used $\frac{2}{2}$ (rather than $\frac{2}{1}$) as an equivalent form of 2, we would not say that this was his most shocking error. It seems more important to note that he failed to notice that multiplying $\frac{1}{3}$ by a number other than one could not possibly result in $\frac{1}{3}$. We would avoid saying "if he had used the correct representation of 2, he would have been able to state that the result is larger." Rather, we would argue that more importantly, accepting his products shows that he believes in his procedure, and so it is not necessary to verify that his answer makes sense. It is interesting to note that when multiplying, this student used $\frac{2}{2}$ as an equivalent form of 2, and when adding used $\frac{2}{0}$. As was the case for other students, equivalence had been reduced to a set of rules for writing equivalent values that came to be far removed from the concept of equivalence. "Equivalence" appeared to be thought of as something you do or make, not as something you maintain.

When concepts atrophy, students are left with no foundation upon which to reason. There were points throughout the interviews where we saw students miss opportunities to reason. One such place was when we asked students to identify the larger of $\frac{a}{5}$ and $\frac{a}{8}$. An answer that involves reasoning might include that some number of fifths is larger than the same number of eighths, if $\frac{a}{5}$ and $\frac{a}{8}$ are based on the same whole. Only two interviewees reasoned in that way. Most students relied on the application of an oft-practiced procedure of creating common denominators, usually $\frac{8a}{40}$ and $\frac{5a}{40}$. Though not incorrect, it is evidence that students apply known procedures rather than using reasoning, even when reasoning is more efficient. (In our previous article we showed that when students used only reasoning about dividing a whole into pieces, the value they identified as larger was always $\frac{a}{5}$.)

In the prior section, we concluded that students rotely apply procedures. The failure we see among them to draw on concepts is the other side of the same coin. For some students we interviewed, basic concepts of number and numeric operations were severely lacking. Whether the concepts were once there and atrophied, or whether never sufficiently developed in the first place, we cannot be certain. What we do know is that these students' lack of conceptual understanding has, by the time they entered developmental math classes, significantly impeded the effectiveness of their application of procedures. And application of procedures is, without concepts and a disposition to reason with them, all that students have left to go on.

What Might We Do to Remedy the Problem?

The goal of much of developmental math education appears to be to get students to try harder to remember the rules, procedures, and notations they've repeatedly been taught. We are thinking about a different solution, one motivated by the picture we've painted of developmental math students. We propose a solution with three elements, each of which is necessary for success. Though in this paper we don't take on the fine details of how to successfully implement the elements in developmental classrooms, we believe that figuring out how to do so might lead to dramatic improvements in student outcomes.

Element One: We must find a way to reawaken students' natural disposition to figure things out and re-socialize them to believe that this is a critical element of what it means to do mathematics. What might such a class look like? Tasks presented to students would be crafted to reveal intuitive understandings of quantity and operations and build number sense, and teachers would create conversations with students to elicit and enhance these understandings. Students' work would highlight the value of mathematical reasoning. They would be pressed to generalize and to consider the limits of their generalizations, sometimes solving problems wholly without the use of procedures. In fact, it would probably be best to pose problems that *cannot* be answered by applying standard procedures, in effect forcing students to *think* to find a solution. If we want students to strengthen their ability to reason productively, we must convince them that such an approach to mathematics can yield them the answers they seek.

Element Two: Necessary to convincing students to think is providing them with productive things to think about. Specifically, rather than asking students to call to memory what they've learned about procedures, ask them to consider the implications of concepts that seem obvious and make those concepts explicit. A teacher might, for instance, connect fractions and division, discussing that a fraction is a division in which you divide a unit into n number of pieces of equal size. Alternatively, the teacher might initiate a discussion of the equal sign, pointing out that it means "is the same as" and not "here comes the answer." Teachers may think that it's implicit in math that concepts, objects, and notation are connected and to point out the obvious would be superfluous, or even demeaning. We argue instead that making big, obvious concepts and connections explicit helps students to organize the domain. The real challenge here, and a place in which further investigation is needed, is to figure out which are the most powerful concepts for students to work with. Which will help connect together the largest portion of the domain?

Element Three: Finally, once students begin to appreciate the value of figuring things out and have begun to lay the foundation of powerful concepts, we can reintroduce procedures into the curriculum. In many cases, standard algorithms map directly onto how students solve problems without them. When this is the case, it should be pointed out. The result will be that procedures are no longer seen as arbitrary, sometimes magical, series of steps, but rather as logical ways to organize effort, connected to core concepts that organize the mathematical landscape. We thus advocate for teachers developing procedures while consistently maintaining connections to the concepts that underlie them. Procedures should be seen by students not as the primary resource available for problem solving, nor as a replacement for thinking. Instead, procedures should be seen as efficient mechanisms for solving problems, supporting and being supported by sense-making.

That we advocate for reasoning and building knowledge of concepts shouldn't be taken to imply that we oppose practice (more commonly associated with learning procedures). Mathematical thinking is a skill and, as such, requires deliberate practice. We suggest that teachers give students repeated opportunities to think and reason, linking core concepts to rules, procedures, and notations. The practice we envision is not one of large numbers of problems similar to each other and to the problem demonstrated by the teacher, but rather small numbers of rich problems carefully selected to highlight and develop concepts and build students' skills in applying them.

Glimmers of Hope

In spite of some of the distressing findings we reported on here and in our prior article, we also see in the interviews some glimmers of hope that the remedies we've suggested may prove effective. It's a nontrivial finding that students were eager to share with us their mathematical thinking and that they required little prompting to do so. Though students are rarely asked to think aloud as they solve problems and to share the rationale for their actions, they quickly fell into that routine. When given an opportunity and limited guidance, (often only the prompt "Why did you do that?") they were able to reason. It took little for us to set those moments in motion: a few, well-crafted prompts, a focus on understanding "why," and a lot of listening to students' thinking. Students also reported having learned from the interview experience (though that wasn't necessarily our intent!). They made connections and saw value in them. Importantly, those experiences of discovering connections were perceived positively—indeed, sometimes joyfully—by students. The key seemed to be to give tasks that presented opportunities to reason and to press students to reason when the opportunity was present.

Student #28 (enrolled in Prealgebra) provides an example of how students sometimes came to reason when encouraged to do so. This conversation shows not only the student's adeptness with procedures, but also willingness to reason directly once he had exhausted the two procedures he knew:

I: If I have these two numbers—I have $\frac{a}{5}$ and $\frac{a}{8}$ — which one of those two is larger?

S: Let's see, $\frac{a}{5}$.

I: Why do you say that?

S: I'm just guessing here, but I got the, I think it's called, the greatest common factor or something, a 5 and 8. So what I just did was I turned the 5 and the 8 into 40, both of them, and I multiplied 5 × 8, so it'd be 8 over 40… 8*a* over 40, and this one would be 5*a* over 40. So I figured that 8 *a*'s is greater than 5 *a*'s.

I: So what you did is you got common denominators, and then you compared the numerators.

S: Yes.

I: Is there a way that you can think about this problem simply by comparing the common numerators as they are?

S: As they are? I guess I could've done $\frac{1}{5}$ and $\frac{1}{8}$. So $\frac{1}{8}$ would be– we could just probably turn them into decimals. So like 5 divided into .10. No. It'll be .2, I guess. And then we turn this one into a decimal, and, well, I think 2, 6, .125 if we turn this one into a decimal.

I: Okay. Do you mind if I write down what you did?

S: Yeah.

I: You turned that $\frac{1}{5}$ into this [.2], and you turned $\frac{1}{8}$ [into .125].

S: Yeah, I guess you can do that. I'm not sure. But it would seem like .2 would still be greater.

I: Yeah. So you've, so far, found two ways to compare them. So you got common denominators and compared the numerators, and then you substituted for *a* and turned it into a decimal. But I'm still curious if you can think about– Is there a way to compare them without doing anything to them?

S: I guess we can, let's say, I don't know. Okay. So let's say $\frac{a}{5}$ and $\frac{a}{8}$ are two pies and they're the same size. If you cut one into 5 pieces and you grab a slice, it'll be bigger than the other pie if you cut it to 8 pieces, and you grab a slice.

I: Great. So three different ways to compare.

Other glimmers of hope came when we explained to students how to use decomposition and the distributive property to do mental multiplication. We demonstrated that 22 × 13 can be thought of as (20 × 13) + (2 × 13) and then asked students to compare that to what they had done when they had solved 13 × 22 using the standard algorithm (for which they normally placed 13 above 22 before they multiplied). Student #8 (enrolled in Elementary Algebra) discovered at this point why there's a "0" in the second partial product of the algorithm, and wondered aloud why he had never noticed it before. By presenting a second method for solving, he gained not only another tool in his repertoire, but also gained a deeper understanding of the standard algorithm. The algorithm was no longer a set of random steps. From this we see that there are many benefits to exploring multiple solution methods and to examining how they relate to one another and to their underlying mathematical ideas. In the interviews, we set up the potential for discovery and even without making a further effort to teach, a student learned. Imagine then, what can happen in a classroom when an effort is made to surface and explore connections.

Directions for Future Work

Our interviews with community college developmental math students cannot be used to fully substantiate our image of where they are and how they might have gotten there. Nor can we know that our suggestions will prove effective. We are, after all, only at the beginning of our journey toward understanding what's going on with these students, mathematically speaking, and how we might be able to change it. We hope that future work will seek to address questions such as whether community college is too late to draw upon students' intuitive concepts about math. Do those concepts still exist? Is community college too late to change students' conceptions of what math *is*? To what degree will students resist a different approach to math teaching and how difficult a task will re-socialization be? (Furthermore, how difficult will be the task of re-socialiazing *instructors*?) If we can change students' beliefs about the nature of mathematics, will it have an effect on their disposition to reason? To what degree does better conceptual understanding impact the successful application of procedures? When students have a disposition and the tools with which to reason, do they apply procedures more appropriately? Do they become bothered by inconsistencies in the results produced by the procedures they apply?

Future work needs also to focus on existing practice in community college math classrooms which, to date, very little work has sought to describe. If there are select places where instructors are emphasizing concepts in their teaching, in what ways and to what degree is it effective? Finally, can some of our suggestions be implemented at the K-12 level and, if so, can we prevent students from having to enroll in community college developmental math classes in the first place?

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Appendix: Questions from the Interview Protocol

Opening

1. If someone is good at math, what exactly are they good at? For example, some people say math is about remembering rules and procedures. Other people say it's about understanding and reasoning. What do you think?

Mental Multiplication

2. 10 × 3 = ______
3. 10 × 13 = ______
4. 20 × 13 = ______
5. 22 × 13 = ______
6. 30 × 13 = ______
7. 31 × 13 = ______
8. 29 × 3 = ______
9. How would you do 22 × 13 it if you weren't asked to do it mentally?
10. Why did you put a "*" [*or '0" or blank*] here?

$$\begin{array}{r} 22 \\ \times\ 13 \\ \hline 66 \\ 22* \\ \hline 286 \end{array}$$

Reverse Operations

$$\begin{array}{r} 462 \\ +\ 253 \\ \hline 715 \end{array}$$

11. How would you check to see if the answer here is correct?
    - *If reworks problem*: Is there another way to check?
    - *If no other way*: Is there a way you can use subtraction to check?

12. Of these two numbers, 572 and 86 [*written horizontally*], which is larger? How do you know?
    - *If 'has more digits'*: Can you always apply that rule? What about 572 and 367?

13. Of these two numbers, 0.572 and 0.86 [*written horizontally*], which is larger? How do you know?
    - *If 'has more digits'*: Can you always apply that rule? What about 0.9 and 0.1111?
    - *If incorrect: correct student and ask if s/he can see why 0.9 > 0.1111.*

14. Can you show me how you would set up 572 – 86? *[written vertically]*

15. Can you show me how you would set up 0.86 – 0.572? *[written horizontally]*
    If incorrectly lined up: Is the placement of the decimals important? How did you decide where the decimals go?

16. Can you show me how you would set up 0.572 – 0.86? *[written horizontally]*
  - *If incorrectly lined up*: Is the placement of the decimals important? How did you decide where the decimals go?

17. Here *[in whole number subtraction]* you lined up the 8 and the 7 and lined up the 6 and the 2. Here *[in subtraction of decimals]* you lined up the 5 and the 8 and lined up the 7 and the 6. Why is that?

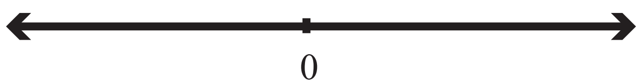
Comparing Fractions and Placing Them on a Number Line

18. Which of these two numbers is larger $\frac{a}{5}$ and $\frac{a}{8}$? How do you know?
  - *If student struggles*: Could you try substituting a number for *a*? Would that be a way to think about it?
  - Will that work no matter what number you choose for *a*?

19. Which of these two numbers is larger $\frac{4}{5}$ and $\frac{5}{8}$? How do you know?
  - *If 4 and 5 are closer together than 5 and 8*: What about these two numbers $\frac{4}{5}$ or $\frac{2}{3}$?
  - *If* $\frac{4}{5}$ *is closer to 1*: Tell me a little more about why that strategy works.

20. Can you draw a number line and place $\frac{4}{5}$ and $\frac{5}{8}$ on it?

    If student is not able to draw a number line, draw just this much:



21. Can you now add these numbers to it $-\frac{3}{4}$ and $\frac{5}{4}$?

a and $\frac{1}{3}$

22. What happens if you take a number and add $\frac{1}{3}$ to it?

23. In this equation $a+\frac{1}{3}=x$, do you think *x* is bigger than *a*, smaller than *a*, equal to *a*, or can you not tell.

24. What happens if you take a number and multiply it by $\frac{1}{3}$?

25. In this equation $a\times\frac{1}{3}=x$, we'll say that a is a positive, whole number. [*Make sure that student understands that '×' means to multiply.*] Do you think *x* is bigger than *a*, smaller than *a*, equal to *a*, or can you not tell?
  - *If student struggles:* What if you had $6\times\frac{1}{2}$?
  - *If student gets it*, return to $a\times\frac{1}{3}=x$.

Closing

26. If you could give math teachers advice about how to teach in a way that would better help you understand math, what would you tell them?