Class #19 - Monday, April 11
Section 5.5: Binomial Experiments, Random Variables & Distributions

HW #6 (due Wednesday, April 18 - day of Exam #2):

- **Section 5.3**: #1, 5, 11, 14, 19, 27
- **Section 5.4**: #1, 4, 5, 10, 14

Definition: A **binomial experiment** is a probability experiment such that:

- The experiment consists of some fixed number n of "trials"
- Each trial has exactly two outcomes (often called success S and failure F)
- The probability of success is the same for each trial
- The trials are independent, i.e., the outcome of any trial does not affect any other

Let the random variable X = the number of successful trials (out of n), which is called a **binomial random variable**. Note that the possible values of X are $0 \leq i \leq n$.

Notation:

- n = number of trials
- p = probability of success in a single trial
- q = probability of failure in a single trial (note that $q = 1 - p$)
- X = the number of successes in the n trials

Example 1: Binomial experiments arise naturally in a number of situations. Understand why each of these is a binomial experiment, and write down the values of n, p and q:

(i) Flip a coin 3 times. Let X = the number of times heads is observed.

(ii) An 80% free throw shooter takes 5 free throws in a game. Let X = the number of successful free throws.

(iii) A bag contains 9 red marbles and 1 blue marble. Fifteen marbles are selected from the bag with replacement. Let X = the number of times a red marble is selected.

(iv) A multiple choice exam has 50 questions. Each question has 5 choices. Suppose you guess randomly on each question. Let X = the number of successful guesses.

(v) Each time a person with a certain infectious disease comes into contact with someone, there is a 5% chance they will pass on the infection. Suppose the person comes in contact with 100 people. Let X = the number of people infected.

We are interested in calculating the distribution of a binomial random variable X, in terms of the parameters n, p and q. Such a probability distribution is called a **binomial distribution**.

Binomial Distribution Formula: In a binomial experiment with parameters n, p and q, the probability of exactly i successes in n trials is:

$$P(X = i) = \binom{n}{i} p^i q^{n-i}$$

Explanation of the binomial probability formula:

- The sample space of a binomial experiment is all 2^n possible sequences of length n of S's and F's
  - Ex: If we flip a coin 3 times, there are $2^3 = 8$ sequences of H's and T's
- Fix $0 \leq i \leq n$. Each different combination of i trials out of the n total trials corresponds to a sequence with exactly i S's. Thus, for each possible value of X, there are $\binom{n}{i}$ sequences with exactly i S's:
  - Ex: There are $\binom{3}{2} = \frac{3*2}{2*1} = 3$ such sequences with exactly 2 heads (namely: HHT, HTH, THH)
- The probability of any one such sequence with exactly i S's (and hence exactly $n - i$ F's) is $p^i q^{n-i}$ (by the Multiplication Rule)
  - Ex: If the probability of heads is p and the proability of tails is $q = 1 - p$, then the probability of HHT is $p * p * q = p^2 q$. Similarly, the probability of HTH is $p * q * p = p^2 q$ and the probability of THH is $q * p * p = p^2 q$.
- Adding up $\binom{n}{i}$ such probabilities results in the product $\binom{n}{i} p^i q^{n-i}$
  - Ex: The probability of getting exactly 2 heads is

$$P(\{HHT, HTH, THH\}) = P(HHT) + P(HTH) + P(THH) = p^2 q + p^2 q + p^2 q = 3p^2 q$$

Example 2: Calculate the following probabilities using the binomial distribution formula:

(i) Take the situation of an 80% free throw shooter who takes 5 free throws in a game. What is the probability that the player will make exactly 3 of the 5 free throws?

(ii) Take the situation of a multiple choice exam with 50 questions, where each question has 5 choices. What is the probability that you correctly guess exactly 40 out of 50 correctly.

Forumlas: The expected value, variance and standard deviation of a binomial random variable X have simple formulas:

- expected value: $\mu = E[X] = np$
- variance: $\mathrm{Var}(X) = npq$
- standard deviation: $\mathrm{SD}(X) = \sqrt{npq}$