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Linear Algebra Sections 1594

Pg.68

31) Let $T : R^n \rightarrow R^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in R^n. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

In exercises 32-36, column vectors are written as rows, such as $\mathbf{x} = (x_1, x_2)$ and $T(x)$ is written as $T(x_1, x_2)$.

If $\{\mathbf{v}1, \mathbf{v}2, \mathbf{v}3\}$ is linearly dependent, then there are x1, x2, x3, not all zero, such that $x1\mathbf{v}1 + x2\mathbf{v2} + x3\mathbf{v3} = \mathbf{0}$. But then

$T(x1\mathbf{v1} + x2\mathbf{v2} + x3\mathbf{v3}) = T(\mathbf{0})$

$T(x1\mathbf{v1}) + T(x2\mathbf{v2}) + T(x3\mathbf{v3}) = 0$

$x1T(\mathbf{v1}) + x2T(\mathbf{v2}) + x3T(\mathbf{v3}) = 0$

Since not all of x1, x2, x3 are zero, this shows that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

33) Show that the transformation T defined by $T(x_1, x_2) = (x_1 - 2x_2, x_1 - 3, 2x_1 - 5x_2)$ is not linear.

$$T\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} x1 - 2x2 \\ x1 - 3 \\ 2x1 - 5x2 \end{bmatrix}$$

Therefore, $T = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & -5 \end{bmatrix}$

Is not linear because, it does not satisfy one of the properties. If T is linear then T (0, 0) = (0, 0, 0)

35) Let $T: R^3 \to R^3$ be the transformation that projects each vector x = (x_1, x_2, x_3) onto the plane $x_2 = 0$, so $T(x) = (x_1, 0, x_3)$. Show that T is a linear transformation.

$u = \begin{bmatrix} u1 \\ u2 \\ u3 \end{bmatrix}$, $v = \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix}$

Prove that : T(u+v) = T(u) +T(v)

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1 \\ 0 \\ u_3+v_3 \end{bmatrix} =$$

$$\begin{bmatrix} u_1 \\ 0 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v}).$$

Prove that: $T(cu) = cT(u)$

$$T(c\mathbf{u}) = T\left(c\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right) = T\left(\begin{bmatrix}cu_1\\cu_2\\cu_3\end{bmatrix}\right) = \begin{bmatrix}cu_1\\0\\cu_3\end{bmatrix} = c\begin{bmatrix}u_1\\0\\u_3\end{bmatrix} = cT\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right) = cT(\mathbf{u})$$

1)

$T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$, and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

$T(e_1) = \begin{bmatrix}3\\1\\3\\1\end{bmatrix}$ $T(e_2) = \begin{bmatrix}-5\\2\\0\\0\end{bmatrix}$

$[\ T(e_1)\ T(e_2)] = \begin{bmatrix}3 & -5\\1 & 2\\3 & 0\\1 & 0\end{bmatrix}$

3)

$T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 3\mathbf{e}_2$, but leaves $\mathbf{e}_2$ unchanged.

$T = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}$

$Te_1 = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix} = e_1 - 3e_2$

$Te_2 = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = e_1 + e_2 = e_2$

5)

$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $\pi/2$ radians (counterclockwise).

T : R^2 – R^2 rotates points (about the origin) through $\frac{\pi}{2}$ radians (counterclockwise).

$$\begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

7)

$T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radians (clockwise) and then reflects points through the horizontal x_1-axis. [*Hint:* $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$.]

$\mathbf{e}_1$ goes to the point $(-1/\sqrt{2}, -1/\sqrt{2})$

e_2 moves to the point $(1/\sqrt{2}, -1/\sqrt{2})$

the point $(1/\sqrt{2}, 1/\sqrt{2})$ after reflection.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

15)

In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

17)

In Exercises 17–20, show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, 0, 2x_2 + x_4, x_2 - x_4)$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

So $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

So T is a linear transform with standard matrix A.