Things I Wish Someone Had Told Me About 1275CO

before I started teaching it!

K. Andrew Parker

August 23, 2019

Contents

PREAMBLE

Before we get started:

1. This is NOT a textbook.
2. This is a text for YOU, the instructor, and NOT for your students.
3. Keeping the first two in mind, please do not distribute the contents of this document.

Each section of this document corresponds to a WeBWorK problem set for MAT1275CO.

The "Prior Knowledge" component provides some notion of what students *should* know before they begin learning the topic. Based on what we expect students to know coming into class, our goal is to build on that prior knowledge as a foundation.

The "Motivating the Topic" component provides suggestions for engaging students in the conceptual purpose for learning the given topic. Sometimes this is based on extending prior knowledge naively – and in so doing, we confront students with some motivation for why our strategies are evolving. Ultimately, we want students to have a meaningful answer for "what's the point of this?"

The "Lesson Suggestions" component tries to offer some possible trains of thought for leading students through the topic. There are frequent recommendations in this component, which are sometimes relevant to topics that come later in the curriculum.

Finally, the "Learning Outcomes" component outlines the desired student learning objectives for the topic. What is it that students should come away with from this portion of the lesson?

UNIT ONE: LINES

Topic: *Lines Review*

Prior Knowledge: Slope formula: $m = \dfrac{y_B - y_A}{x_B - x_A}$
Point-Slope form: $y - y_A = m(x - x_A)$ or $y = m(x - x_A) + y_A$
Slope-Intercept form: $y - y_0 = m(x - 0)$ or $y = mx + y_0$

Lesson Suggestions: Students frequently misuse formulas that rely on numeric subscripts such as x_1 and x_2

Try using x_A and y_A to refer to the coordinates for point A: (x_A, y_A).

This set should be comprised entirely of review content.
Very little instruction required, emphasize in-class practice. Establish your classroom as a place where students are asked to work in groups (from time-to-time, or more often!)

Use this time to get your students comfortable with logging into WeBWorK.

Learning Outcomes: Students construct equations for lines based on a combination of given points and/or slope.

Topic: *Graphing Lines*

Prior Knowledge: Slope formula: $m = \dfrac{y_B - y_A}{x_B - x_A}$
Point-Slope form: $y - y_A = m(x - x_A)$ or $y = m(x - x_A) + y_A$
Slope-Intercept form: $y - y_0 = m(x - 0)$ or $y = mx + y_0$
Standard form: $Ax + By = C$
Parallel lines have the same slope.
Perpendicular lines have slopes that are reciprocals with opposite sign.

Learning Outcomes: Given a linear equation (in any form):

- Identify the slope
- Identify the y-intercept
- Identify the x-intercept
- Graph the line

Given a point and a slope:

- Graph the line
- Give an equation for the line
- Identify the intercepts for the line

Given a line and a point not on the line:

- Graph the line parallel to the given line, passing through the given point
- Identify the slope of the given line and infer the slope of the parallel line
- Give an equation for the parallel line passing through the given point
- Graph the line perpendicular to the given line, passing through the given point
- Identify the slope of the given line and infer the slope of the perpendicular line
- Give an equation for the perpendicular line passing through the given point

This section ends with finding a point of intersection by graphing two given lines. (First in slope-intercept form, and then in standard form.)

Topic: *Line Lab*

Prior Knowledge: For this task, the following are helpful, but not required. Groups of students will most likely contain at least one student with awareness of each of the following:

Slope formula: $m = \dfrac{y_B - y_A}{x_B - x_A}$

Point-Slope form: $y - y_A = m(x - x_A)$ or $y = m(x - x_A) + y_A$

Slope-Intercept form: $y - y_0 = m(x - 0)$ or $y = mx + y_0$

Standard form: $Ax + By = C$

Lesson Suggestions: This task is intended as a discovery activity.

Problem one serves as an introduction to the platform and the interactive nature of the graph applet. Students drag the points until the graph matches the equation given.

Problem two is intended to challenge students to create their own lines and then give the equation of the line they have graphed. This task must be completed in four unique ways before full credit is given. Students are encouraged to simply make attempts (right or wrong) in coming to terms with the connections between the intercepts and the equation for the line.

To do: Add a worksheet and/or guiding questions to facilitate this activity

Learning Outcomes: Students will formulate algebraic equations from a line created by x- and y-intercepts.

Topic: *Linear Systems*

Prior Knowledge: In "Graphing Lines," students began finding the intersection of two lines by graphing the given linear equations. They should have seen this process for two lines in slope-intercept form as well as in standard form. Students should be asked to identify which forms are most conducive to finding the intersection by the graphing method. (Assume: students will find slope-intercept to be the easiest form for graphing approach.)

Motivating the Topic: Give students a system of linear equations in slope-intercept form where the point of intersection has at least one non-integer coordinate.

Lesson Suggestions: Students should see both substitution and elimination methods.
Substitution is most relevant to begin the lesson, building off students' (presumed) preference for equations in "$y =$" form.
Substitution can also be shown to be relevant when only one (instead of both) equation is given with y isolated.
Substitution can also be shown to be relevant when x is isolated instead.
Substitution can be shown to be more difficult when neither x nor y may be "easily" isolated (i.e. isolating any variable requires work with fractions) .

Elimination method should somewhere include discussion of the additive property of equality when what's added to both sides of the equation is equivalent, though not identical:

$$10 = 10$$

$$10 + (1 + 1) = 10 + 2$$

because $(1 + 1) = 2$

Learning Outcomes: Students understand that the "solution" to a system of equations is a coordinate-pair that satisfies both equations.
Students understand that the "solution" to a system of 2-D linear equations corresponds to the point where the graphs of the two lines intersect.

Topic: *3x3 Systems*

Prior Knowledge: Elimination method in the 2x2 case for linear systems

Motivating the Topic: Eliminating one variable (for the first time) is easy.
e.g. eliminate x in a 3x3 system by combining two equations.
Show that using the third equation, we can try to eliminate one of the remaining variables - but in so doing, we re-introduce x into the equation.

Lesson Suggestions: We can create a 2x2 problem from a 3x3 problem by selecting a variable to be eliminated (and then eliminating it twice – using two different combinations).
Students gravitate towards variables that already have the same coefficients (or better, coefficients with opposite signs and the same absolute value). This has immediate value, but the requirement to eliminate the same variable with two different combinations pushes us to look for a deeper strategy. Redirect students attention to variables with coefficient ± 1. Ones act like "wild cards" that can be easily transformed into any other value - providing a quick elimination with both of the other equations.

It can be useful to have students label each equation with a letter. Equation "A", "B", and "C", for example. Then eliminations can be formally expressed as, for example, $A + (-2)B$. Resulting equations of two variables should also then be labeled, "D" and "E" are an obvious extension.

It is important for students to learn to check their results.
Students should also experience problems where their answers are not entirely comprised of integer values.

Learning Outcomes: Students understand that 3x3 systems of linear equations are an extension of 2x2 systems.
Students understand that their solution consists of values that play the role of coordinates just as in the 2x2 case, except now we have 3 coordinates instead of 2.

UNIT TWO: QUADRATICS

Topic: *GCF Grouping*
GCF factoring & Factoring by Grouping

Prior Knowledge: Distributive property: $a(x+y) = ax + ay$
GCF for integers
Associative property of addition
Rewriting differences as sums

Motivating the Topic: Recognizing binomials (polynomials) where the distributive property might have been used – and then writing what the expression looked like *before* the distributive property was used.
Similarly, the recognition that adding (or subtracting) multiples of the same value results in another multiple of the same value. For example, two multiples of 5: $15 + 35$ add up to: 50, another multiple of 5, and that $5 \cdot 3 + 5 \cdot 7$ gave us $5 \cdot 10$.
This works for all *numbers*, so it works for algebraic expressions also.
This is the basis for "like terms": $3x + 7x$ is the sum of two multiples of x, so: $(3+7)x$ or $10x$

Lesson Suggestions: Main focus: binomial GCF factoring, have students do many - they're quick.
Special example: factoring out the *negative* GCF (from sums and differences).

For grouping, we need the associative property of addition.
Emphasize the need for sums when grouping (some students will be able to consistently perform grouping without this step, but many students at least need this as training wheels).
Grouping examples should be the result of multiplying binomials that do not contain like terms; for example: $2AB + 3B + 4A + 6$ or $x^3 + 5x^2 + 10x + 50$.
It is important to show at least one non-example where grouping fails (i.e., the resulting binomial factors do not match - *then what?*)

Learning Outcomes: Students can identify the GCF of a polynomial and apply the distributive property to rewrite it as a product.
Students can apply the associative property of addition and GCF factoring to a four-term polynomial in order to write it as the sum of binomials.

Topic: *Difference of Squares*

Prior Knowledge: Factoring by Grouping

Motivating the Topic: The result of factoring a difference of squares is very formulaic - but take the time to at least show how the prior topic of factoring by grouping ties into this topic.
$x^2 - 49$ is a difference of squares.
x is squared and 7 is squared.
$x^2 - 7x + 7x - 49$ is equivalent to the expression we were given.
Apply factor by grouping, and see how this factors.

Lesson Suggestions: Using the distributive property on anything with the form $(A + B)(A - B)$ always results in a difference of squares.
Identify A and B, then conjecture the form of the product that results in $A^2 - B^2$.
Verify that the product works out to the expression that is expected.

Learning Outcomes: Students will recognize differences of squares as a special category of binomials.
Students will recognize the (conjugate) relationship between the two factors of a difference of squares.

Topic: *AC Method*

Prior Knowledge: Factoring by Grouping

Motivating the Topic: Many students have already seen the "start with the parenthesis" approach to this topic (and yet they placed into this class).
We intend to show students the full process of reversing the distributive process for multiplying two binomial factors.
Show all steps for finding the product of two binomials:

$$(x + A)(x + B)$$

$$x(x + B) + A(x + B)$$

$$x^2 + Bx + Ax + AB$$

$$x^2 + (A + B)x + AB$$

Lesson Suggestions: Work the above steps in reverse, for a particular example: $x^2 + 8x + 7$
Recognize that we have values for A and B (students are often familiar with this process, but may not be completely clear as to why we should care).
rewrite $+8x$ as $+1x + 7x$: $x^2 + 1x + 7x + 7$
and then factor by grouping.
Show students what happens if they "split" the linear term in the reverse order:

$$x^2 + 7x + 1x + 7$$

This is relevant because the GCF in the second group is 1 (which should be specifically addressed).

Some students will already be adept at finding binomial factors for a quadratic with leading coefficient 1 (monic). They should be required (with everyone else) to show the intermediate steps at least for the duration of this lesson. The intent is that they practice the AC-method on the "easier" monic quadratics so that they are familiar with the thought process when we next move on to non-monic quadratics.

Guess-and-check is not recommended for non-monic quadratics.

Teach students the AC-method:
Find A, B, and C. Compute $A \cdot C$. Identify the two values whose sum is the linear coefficient and whose product is the constant term. Proceed by splitting the linear term and then factoring by grouping.

Learning Outcomes: Students will understand the process of factoring quadratic polynomials, both monic and non-monic.

Topic: *Zero-Product Property*

Prior Knowledge: Methods of factoring: GCF, grouping, AC-method

Motivating the Topic: Solving an equation requires isolating the variable, but when we have a quadratic equation - the three distinct "un-like" terms prevent us from isolating the variable. (This can be avoided by focusing on two-term difference-of-squares quadratics, in which case you might want to swap the order of this topic with Square-Root Property.)

Zero has an interesting relationship with multiplication. "zero times anything (finite) is zero."
BUT, can *anything else* give us zero after multiplying?
In other words: $A \cdot B = 0$, what could A or B be?

Lesson Suggestions: $A \cdot B = 0$ if *and only if* at least one of A or B is zero. This is called the zero-product property.
Factoring only makes sense as a strategy if we can say the product is zero.
If the product is something non-zero, there are *too many* possible factorings: $AB = 5$ then $A = 2$ and $B = \frac{5}{2}$, or $A = n$ and $B = \frac{5}{n}$, etc. We can't pin down either A or B.

By the end of the homework, students are solving non-monic quadratic equations by factoring.
Students should be able to discern that solutions to quadratic equations are *not quite* the same as the values that are found during the AC-method process.

Learning Outcomes: Students will be able to apply the AC-method to solve quadratic equations, both monic and non-monic.

Topic: *Square-Root Property*

Prior Knowledge: The process of simplifying square-root expressions with integer radicands; The value of square-root expressions as irrational values with only approximated decimal representations; The common structure of expanded perfect-square-binomials;

Motivating the Topic: This lesson should illustrate the difference between sentences such as $x = \sqrt{9}$ and $x^2 = 9$.
$\sqrt{9}$ refers only to the value 3, whereas $x^2 = 9$ has two possible solutions - both the positive and negative versions of "the number whose square is 9" (as represented by $\sqrt{9}$).
so $x = \pm\sqrt{9}$ in the second case.

Motivating the use of completing the square requires us to ask students to solve quadratic equations that have irrational and/or imaginary solutions.

Lesson Suggestions: Take the time to be explicit about the intended meaning behind the shorthand notation of plus-minus ($\pm$). For example, $x = \pm 3$ is actually two equations: $x = +3$ and $x = -3$.

Difference-of-Squares equations make for easy examples, be sure to use some non-perfect squares as well.
Try a sum-of-squares and see that the result requires the square root of a negative value.
Introduce the imaginary number.
Students do not need to know more than that $i = \sqrt{-1}$ at this point. There is no *Real* number with the property that its square is -1, therefore it must be "imaginary" (since it is not real).
Students should be familiar with the property of square-roots: $\sqrt{AB} = \sqrt{A}\sqrt{B}$; and it is only for us to explain that we apply this property to factor out the square-root of -1 which then simplifies to i.

Returning to the square-root property:
If we are given the result of a squared-expression, we can determine the value that was squared, but not its sign. In other words: $\square^2 = A$,where $\square$ is any algebraic expression, then $\square = \pm\sqrt{A}$.

Apply to an explicit example, such as: $(x-2)^2 = 16$
If there is time, square the binomial and solve using zero-product property (provided that you've already covered it). Show students the consistency in solutions, regardless of approach.

Show the result of squaring any monic, linear binomial: $(x+C)^2$ as taking the form $x^2 + 2C + C^2$.
It may help to use a visual aid, such as the box-method for multiplying out this square. (It will help when completing the square to have a similar visual aid.)

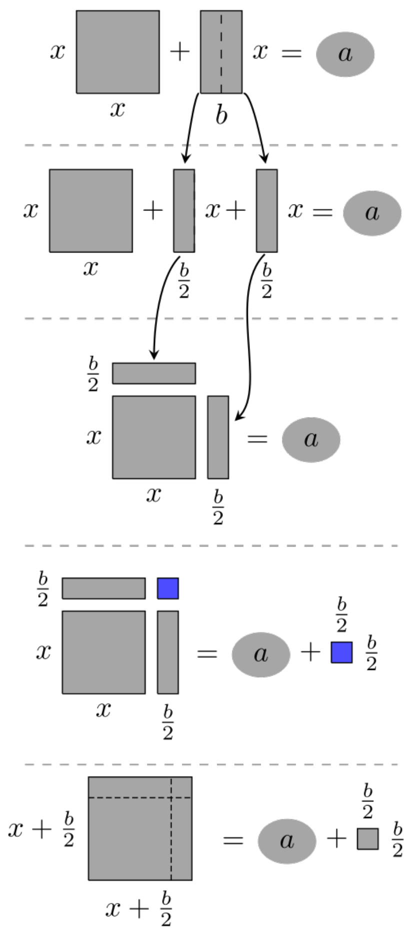
You may wish to use a specific value for b and a in the provided example.

Ask students to identify the missing value in expressions such as:
$x^2 + 10x + \_\_$
or $x^2 - 16x + \_\_$
and then lead into the process of completing the square. (The illustration here may be useful.)

This lesson may be easily blended with the lesson for Quadratic Formula, in case time runs short.

Completing the Square

$$\boxed{x^2 + bx = a}$$



$$\left(x + \tfrac{b}{2}\right)^2 = a + \left(\tfrac{b}{2}\right)^2$$

$$\boxed{x = -\tfrac{b}{2} \pm \sqrt{a + \tfrac{b^2}{4}}}$$

Learning Outcomes: Students will be able to apply the process of completing the square to solve <u>monic</u> quadratic equations in the cases where solutions are irrational and/or imaginary.

Topic: *Quadratic Formula*

Prior Knowledge: Square-Root Property
Completing the Square
Definition of i

Motivating the Topic: What about *non-monic* quadratics that do not admit a zero-product property solution?

Apply the additional steps to turn a non-monic complete the square problem into a monic one.

Lesson Suggestions: Apply completing the square to a non-monic quadratic equation with "nice" solutions, just to make the introduction to the "extra steps" a little easier. For example: $2x^2 - x = 6$

Apply completing the square to a non-monic quadratic equation with irrational solutions.

Apply completing the square to the fully-abstract case: $Ax^2 + Bx + C = 0$, and end up with the quadratic formula.

The point of emphasizing the quadratic formula is to circumvent all the steps of completing the square when we find ourselves in a situation where zero-product property is not available.

The discriminant, $B^2 - 4AC$, indicates the nature of our solutions:

- $B^2 - 4AC > 0$ indicates that there will be two real solutions
- $B^2 - 4AC = 0$ that there will be only one real solution
- $B^2 - 4AC < 0$ that there will be two imaginary (complex) solutions

It is up to the instructor to decide when to broach the topic of "imaginary" versus "complex".
The assignment for this session does not require students to enter their answers in "$a + bi$" form.

Learning Outcomes: Students will learn to apply the quadratic formula to the task of solving quadratic equations when necessary.
When applying the quadratic formula, students will attend to precision when simplifying.
Students will understand that the discriminant will determine the nature of the solutions to the quadratic equation: two real solutions, one real solution, or two imaginary (complex) solutions.

Topic: *Complex Numbers*

Prior Knowledge: Definition of i as "the number whose square is -1"

Motivating the Topic: Review the definition of i and discuss its powers.

Lesson Suggestions: We never need any power beyond i^1.
Discuss the cyclical nature of powers of i.
Analogy: Clocks "start over" after reaching 12 (or 60).

Operations with complex numbers:
Complex numbers have two parts, real and imaginary.
Students should see "i" as behaving like a variable - but instead, i has a very specific value that allows us to simplify any power of i.

Complex numbers have "conjugate" pairs. Specify that conjugate pairs have *identical real parts*, while their *imaginary parts differ only in sign.*

Learning Outcomes: Students should be able to apply the four primary operations to complex numbers and express their result as a complex number (in $a + bi$ form).

Students should understand the cyclical nature of the powers of i and apply them to simplify any power of i.

Topic: *Shifting Parabolas*

Prior Knowledge: The coordinate plane and graphing, generally
The shape of $y = x^2$ as a parabola with vertex at the origin

Motivating the Topic: We try to set the stage for vertex form of a parabola.
Students should see the effect of changing parameters in both $y = x^2 + k$ and $y = (x - h)^2$
Desmos is an easy way to display these two forms and the relevance of the parameters h and k

Lesson Suggestions: Very little to say here, because students **cannot** be assumed to have familiarity with the functional context. Discussing functions here risks losing a significant proportion of the class.

Learning Outcomes: Students should understand that "modifying x before squaring" (in $y = x^2$) has the effect of translating the parabola horizontally.
Students should understand that "modifying x after squaring" (in $y = x^2$) has the effect of translating the parabola vertically.

Topic: *Parabola Vertices - Completing the Square (CtS)*

Prior Knowledge: Completing the Square
The shape of $y = x^2$ as a parabola with vertex at the origin
The effects of translating parabolas both horizontally and vertically
How to manipulate $y = x^2$ in order to achieve a specific translation.

Motivating the Topic: This is presumed to be covered along with "Shifting Parabolas", so we ought to connect our horizontal and vertical translation to the change in the position of the vertex of the parabola.
Introduce vertex form of a parabola: $y = (x - h)^2 + k$, noting the signs on h and k (and connecting back to their effect via translation on the vertex).

Lesson Suggestions: In order to convert a quadratic equation (in x and y - before we only had x!) into vertex form, we need to apply the process of completing the square.
Reference: $(x - h)^2$ as a perfect square.
This time, however, we have y to contend with (instead of zero - or some other constant). The homework walks students through the process step-by-step. At first for monic parabolas, and eventually for non-monic ones as well.

Learning Outcomes: Students will apply the process of completing the square to the equation of a parabola in order to express it in vertex form.
Students will interpret vertex form in order to provide the vertex (as a point) for the given quadratic equation.
Students will understand vertex form of a parabola as resulting from the vertical and/or horizontal translation of a parabola starting from a vertex at the origin.

Topic: *Parabola Vertices - Vertex Formula*

Prior Knowledge: Quadratic Formula
Symmetry of the parabola

Motivating the Topic: Completing the Square is undesirable (as a time-consuming process).
Identify the horizontal symmetry of the parabola, as well as the vertex's unique position ON the line of symmetry.

Lesson Suggestions: "What points do we always know how to find on our parabola?" –¿ "the roots."
Our roots are always symmetric, and their midpoint gives us the vertical line of symmetry.
The vertical line of symmetry gives us the x-coordinate of our vertex.
The equation gives us the y-coordinate once we know the x coordinate.

The above could be done with a corresponding specific quadratic to work on, or could be done without discussing the equation at all.
Regardless, it should be now addressed that our roots are provided by the quadratic function, and the average of the two roots will always be $-\frac{B}{2A}$.
Use this to justify the entirety of the vertex formula.

Learning Outcomes: Students will understand the origins of the vertex formula as stemming from the symmetry of parabolas in combination with the quadratic formula.
Students will apply the vertex formula and provide the coordinates of the vertex for any given quadratic equation.

Topic: *Distance Formula*

Prior Knowledge: This is a review topic. Students should already know the distance formula.
Pythagorean Theorem

Motivating the Topic: The linear distance between two points can be found from a combination of their horizontal and vertical distances.

Apply Pythagorean Theorem.

Lesson Suggestions: point A: (x_A, y_A) and point B: (x_B, y_B)
horizontal distance $= |x_B - x_A|$
vertical distance $= |y_B - y_A|$
(absolute values because, unlike slope, we don't care about which way is up/down/left/right).
vertical and horizontal distances form two legs of a right triangle with linear distance as the hypotenuse.

Learning Outcomes: Students will apply Pythagorean Theorem to determine the distance between two points given as coordinate pairs.

Topic: *Circles*

Prior Knowledge: Distance Formula
Pythagorean Theorem
Vertical and Horizontal translation
(Students frequently remember $x^2 + y^2 = r^2$ as the equation of a circle)
Completing the Square

Motivating the Topic: Circles are defined as "all the points having a fixed distance from a given center point"
How do we say that algebraically?

Lesson Suggestions: given a center point, C: (x_C, y_C)
and a radius: r
how do we denote "all the points" from the definition above?
an unspecified point: (x, y) might be on our circle or it might not be.
This is okay because that's always the case, we have points that satisfy our equation (on our curve) and points that do not satisfy our equation (not on our curve).

so the distance from the center: $d = \sqrt{(x - x_C)^2 + (y - y_C)^2}$, must be a fixed distance: $= r$

in other words: $\sqrt{(x - x_C)^2 + (y - y_C)^2} = r$
or
$(x - x_C)^2 + (y - y_C)^2 = r^2$

If students already recognize $x^2 + y^2 = r^2$ as the "circle formula", then we should also address the vertical and horizontal translations (being consistent with translations of parabolas just a few topics back).

Students now must complete the square and convert circles from expanded form to standard form.
Reminder: this is why students practiced completing the square in earlier topics - it's an unavoidable technique here!

Learning Outcomes: Students will use the standard form to provide an equation for a circle, given a center point and a fixed radius.
Students will apply the process of completing the square to convert the equation of a circle into standard form.
From an equation of a circle in standard form, students will infer the location of the circle's center and its radius.

Topic: *Non-Linear Systems*

Prior Knowledge: Solving 2x2 linear systems by elimination
Methods for solving quadratic equations

Motivating the Topic: What happens when we let *curves* intersect instead of just lines (as was previously the case in 2x2 systems)

How many ways might a line intersect a parabola?
A line and a circle?
A circle and a parabola?

Lesson Suggestions: This lesson will benefit greatly from visual representations (such as provided by Desmos).

Introduce students to the means by which we can classify these curves based on their equations.

- Neither x nor y "squared" means this is a linear equation.
- Either x OR y "squared" **but not both** means this is parabolic.
- Both x AND y "squared" means this is either elliptical or hyperbolic (desmos)

Students should be asked to think about the ways in which these shapes can intersect one another (emphasizing the inherent symmetries!)

Overall strategy is:

- substitution whenever a convenient line is present;
- elimination of the quadratic term in the case where only one variable appears as squared across both equations;
- elimination as in the usual 2x2 linear case whenever no linear terms are present

Verification of all solutions should be emphasized, as well as asking "does the number of solutions make sense based on the shapes being intersected?"

Learning Outcomes: Students will classify the shapes of graphs given by various quadratic equations in two variables.
Students will infer the possible number of intersection points allowed by various combinations of quadratic graphs.
Students will select and apply an appropriate strategy in finding intersection points of quadratic graphs.

UNIT THREE: RATIONAL ALGEBRA

Topic: *Reducing Rational Expressions*

Prior Knowledge: Reducing numeric fractions
Various means of factoring: GCF, AC-method

Motivating the Topic: Just as numeric fractions can be expressed in multiple ways (with a unique "reduced form"), so can algebraic rational expressions.
Both are "reduced" by considering common factors of the numerator and denominator. For example:
$\frac{14}{35} \rightarrow \frac{2 \cdot 7}{5 \cdot 7} \rightarrow \frac{2}{5} \cdot \frac{7}{7} \rightarrow \frac{2}{5} \cdot 1$

Lesson Suggestions: The same applies to monomial fractions:
$\frac{x}{x^3} \rightarrow \frac{x}{x \cdot x \cdot x} \rightarrow \frac{1}{x \cdot x} \cdot \frac{x}{x} \rightarrow \frac{1}{x^2} \cdot 1$
Continue applying to binomial numerator / monomial denominator and vice-versa (monomial / binomial)
Then get into reducing rational expressions with quadratic numerators and/or denominators.

It is not too early to begin pointing out the non-trivial GCF between binomials such as $x - 5$ and $5 - x$. This will be relevant in later sections.

Learning Outcomes: Students will understand the relevance of GCF in reducing rational expressions

Topic: *Add Rational Expressions (1 and 2)*

Prior Knowledge: Adding (or subtracting) numeric fractions requires a common denominator
Common multiples (preferably "least")
Reducing rational expressions

Motivating the Topic: Algebraic fractions are hardly different from numeric fractions.

Lesson Suggestions: Sums and differences require a common denominator (which is present in the first problem set), and the only difference is in subtracting algebraic fractions. For example:

$$\frac{5}{x+1} - \frac{x+10}{x+1}$$

It is important to empasize the *implicit* parenthesis that are present in fractions:

$$\frac{5}{x+1} - \frac{x+10}{x+1} \rightarrow \frac{(5)-(x+10)}{(x+1)}$$

For the second problem set, students must convert to a common denominator before adding or subtracting. Some discussion regarding *least* common multiples is called for. For instance, if students do not find the least common denominator, they will certainly cause the result to be reduce-able.

Regardless, the least common denominator is never explicitly required - though final answers are traditionally required to be fully reduced.

By the end of this lesson, students should have been at least somewhat familiarized with the GCF of expressions such as $x - a$ and $a - x$.

Learning Outcomes: Students will convert fractions to a common denominator in order to add and subtract rational expressions.

Topic: *Complex Fractions: invert-and-multiply (method 1)*

Prior Knowledge: Multiplication and division of numeric fractions
Reducing rational expressions

Motivating the Topic: Complex fractions are just fractions-of-fractions, or in other words: division of expressions containing fractions

We know how to divide fractions, so all that needs to be done is to *combine* the numerator's fractions into a single rational expression, and then also combine the denominator's fractions.

The rest is handled by applying the known-strategy of division as *multiplication by the reciprocal*.

Starting with an entirely numeric example can be useful to avoid scaring students who are skittish about fractions.

Lesson Suggestions: Apply the lessons on adding fractions with un-like denominators in order to combine the numerator into a single fraction. Similarly combine the denominator.

It may be useful to choose a word to describe the "outer" fraction in contrast to the "inner" fractions (or "largest" vs. "smaller", etc.) - this terminology will draw students attention to the various levels of fraction that we are working to simplify, and hopefully prevent confusion from saying "fraction" all the time.

Students frequently recall the process of multiplying by the reciprocal, but they often do not know why they are doing it...

A quick justification:

$$\frac{\frac{A}{B}}{\frac{C}{D}} \to \frac{\frac{A}{B}}{\frac{C}{D}} \cdot \frac{\left(\frac{D}{C}\right)}{\left(\frac{D}{C}\right)} \to \frac{\frac{A}{B} \cdot \frac{D}{C}}{\frac{CD}{CD}} \to \frac{\frac{A}{B} \cdot \frac{D}{C}}{1} \to \frac{A}{B} \cdot \frac{D}{C}$$

Learning Outcomes: Students will convert division into multiplication in order to simplify complex fractions.
This technique will correlate to one of the strategies for solving fractional equations.

Topic: *Complex Fractions: Least Common Multiple (method 2)*

Prior Knowledge: Multiplication of fractions
Least Common Multiples

Motivating the Topic: Multiplication has the potential to reduce fractions to a denominator of 1 (at which point, we no longer require fractions to represent them).

Fractions-of-fractions may be multiplied by any expression of the form $\frac{\square}{\square}$.

Combine these two ideas to reduce complex fractions.

Lesson Suggestions: Multiply both the numerator and denominator of the "outermost" (or "largest") fraction by the LCD of all the "inner" (or "smaller") fractions. This will "clear the denominators" in each of the "smaller" fractions.

Simplifying a complex non-algebraic fraction in this manner can be useful to acclimate students to the strategy (without the added cognitive load of identifying the LCD of algebraic fractions).

Learning Outcomes: Students will apply the strategy of multiplying by the $\frac{LCD}{LCD}$ in order to reduce complex fractions.
This technique will also correlate to one of the strategies for solving fractional equations.

Topic: *Fractional Equations - solving rational equations*

Prior Knowledge: Adding and subtracting fractions
Multiplying by the LCD to "clear denominators"

Motivating the Topic: Use the techniques we've learned to solve equations involving rational expressions.

Rational equations are weird, because $\frac{2}{3} = \frac{4}{6}$ even though neither the numerator nor denominator are actually equal...

Lesson Suggestions: Two major strategies arise (mimicking the complex fraction strategies):

- Convert all fractions to have the same denominator, in which case, the numerators satisfy the equation.
- Multiply both sides of the equation by the LCD in order to "clear denominators" and leave a non-rational equation to solve.

In either case, students *must*, at least, check their answers to ensure they do not cause division by zero.
Extraneous solutions will appear again in the radical equations section - it is worth it to begin the conversation here.

In "clearing denominators" students may be multiplying both sides of the equation by zero...

By "ignoring denominators" students may also be ignoring a zero denominator...

Learning Outcomes: Students will employ multiple strategies in solving rational equations.
Students will understand the manner in which extraneous solutions can arise from the process of solving rational equations.

UNIT FOUR: EXPONENTS AND RADICALS

Topic: *Integer Exponents*

Prior Knowledge: Positive integer exponents
Scientific Notation
Many students will have prior awareness of the exponent properties as well

Motivating the Topic: Making sense of exponents - many students "know" the properties, but might not understand why they work.

Exponents are an abbreviation for repeated multiplication by the base - but what about negative exponents?

Lesson Suggestions: Write out the simplification of various exponential properties longhand, rather than skipping straight to the "rule".

Write out what it *means* for x to be raised to the 5^{th} power, etc.

Write out what it means to have $5x^2$ raised to the third power - *reinforce* the dirty work that gets swept under the rug when we apply the properties.

Learning Outcomes: Students understand how the exponential properties arise from abbreviating repeated multiplication as exponents.
Students understand the existence of negative exponents as multiplication of the reciprocal of the base.

Topic: *Higher Roots*

Prior Knowledge: Computing positive powers of integers and rational numbers

Motivating the Topic: "Square-root" of a number refers to "the number whose square" is equal to that number

We can use the same basic structure to create "cube-root" and other radicals with higher index.

Lesson Suggestions: It is productive to take the time to define the terms: radical, radicand, and index.

Students might need an explanation for the multiplicative property for square-roots (which we can extend to all radicals):
$\sqrt{A}$ refers to "the number whose square is A" ($\sqrt{A} \cdot \sqrt{A} = A$)
$\sqrt{B}$ refers to "the number whose square is B" ($\sqrt{B} \cdot \sqrt{B} = B$)
Then $\sqrt{A}\sqrt{B}$ satisfies: $\sqrt{A}\sqrt{B} \cdot \sqrt{A}\sqrt{B} = AB$,
So, in other words, $\sqrt{A}\sqrt{B}$ has AB as its square - meaning it is the square-root of AB ($\sqrt{A}\sqrt{B}$ is the same as $\sqrt{AB}$)

"Root" is caught in a weird spot here (as a vocabulary word) - it simultaneously refers to solutions to our earlier (quadratic) equations, and now here it means something else.

A major theme for this introductory lesson is the fundamental difference between radicals of even-index and radicals of odd-index (and their effect on negative radicands).

Learning Outcomes: Students will interpret the meaning of a radical expression with index other than 2.
Students understand the different results that occur when taking even or odd indexed radicals of negative values.

Topic: *Higher Roots - Algebraic*

Prior Knowledge: Radical expressions with index other than 2

Motivating the Topic: How does our simplification of radical expressions change when we are uncertain of the specific value in the radicand?

Lesson Suggestions: The major theme here is that even-indexed radicals cannot result in a negative value. Students must learn to interpret this situation and express it in terms of absolute values.

This detail is not reinforced throughout the remaining lessons - instead, after discussing and assessing this topic, radical problems should provide disclaimers: "assuming all variables represent positive values..." (or something similar).
It is worth pointing out this change to students as well (after you've moved on from this lesson).

Learning Outcomes: Students recognize when radicals do not simply "cancel" existing exponents; and they express simplified radical expressions accurately with regard to absolute values.

Topic: *Rational Exponents*

Prior Knowledge: Integer Exponents
Exponent Properties
Radicals with Index

Motivating the Topic: "If radicals could be expressed as exponents, what value would that exponent be?"
$\sqrt{x}$ is "the number whose square is x", so $\sqrt{x} \cdot \sqrt{x} = x$"
We do not know the value of the exponent that corresponds to "square-root", use a.
Now: $x^a \cdot x^a = x$ and x is just x^1, so now have: $x^{2a} = x^1$
We must conclude that $2a = 1$ or $a = \frac{1}{2}$

Repeat for "cube-root", etc.

Lesson Suggestions: With enough examples (as above), we recognize the pattern that $\sqrt[n]{x}$ can be expressed as an exponent: $x^{\frac{1}{n}}$

What about rational exponents that have a numerator *other than* 1?

$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
So: $x^{\frac{3}{4}} \rightarrow x^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} \rightarrow x^{\frac{1}{4}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{4}} \rightarrow \sqrt[4]{x} \cdot \sqrt[4]{x} \cdot \sqrt[4]{x} \rightarrow (\sqrt[4]{x})^3$

We could instead apply the multiplicative property of radicals:
$\sqrt[4]{x} \cdot \sqrt[4]{x} \cdot \sqrt[4]{x} \rightarrow \sqrt[4]{x^3}$

This topic also covers the difference between writing exponential expressions with and without parenthesis around a negative base.

Students must also learn to integrate their understanding of negative exponents with rational exponents. It is common for students to apply the negative exponent as a reciprocal *of the rational exponent* instead of applying it to the base as they ought to.

Learning Outcomes: Students will translate expressions between rational exponent-form and radical-form

Topic: *Simplifying Radicals: square-roots only*

Prior Knowledge: Simplifying square-roots of integers
Rational Exponents

Motivating the Topic: Integers: $\sqrt{98} \to \sqrt{2 \cdot 7 \cdot 7} \to \sqrt{2} \cdot \sqrt{7} \cdot \sqrt{7} \to \sqrt{2} \cdot 7$

Lesson Suggestions: It's exactly the same when it comes to variables:
Variables: $\sqrt{x^3} \to \sqrt{x \cdot x \cdot x} \to \sqrt{x} \cdot \sqrt{x} \cdot \sqrt{x} \to x \cdot \sqrt{x}$

Note: This topic explicitly uses variables that are restricted to positive values.

Some students find it useful to think of reducing algebraic square-roots via rational exponents.
$\sqrt{x^7} \to x^{\frac{7}{2}}$ and then we think of turning $\frac{7}{2}$ into a mixed fraction (or decimal). 3.5 or $3 + \frac{1}{2}$, either way, we simplify to $x^3\sqrt{x}$.

By the end of this lesson, students should be able to simplify an algebraic square-root expression with an algebraic coefficient.

Learning Outcomes: Students will simplify monomial, algebraic, square-root expressions under the condition that all variables represent only positive values.

Topic: *Adding & Subtracting Radicals*

Prior Knowledge: Simplifying Radicals
Distributive Property / "Like Terms"

Motivating the Topic: $3x + 5x$ is $8x$
Can x be any number? What if x is $\sqrt{7}$?
Then: $3\sqrt{7} + 5\sqrt{7}$ is $8\sqrt{7}$

Lesson Suggestions: Review "like terms" as a specific way of using the distributive property. It applies to common square-root factors just as it does for variables.

There's not much to cover here, problems rely on simplification of square-root expressions *first*, and then identifying "like terms" according to any remaining square-root factors.

Learning Outcomes: Students will apply the process of simplifying square-root expressions in identifying and combining compatible terms from a given sum or difference.

Topic: *Multiply Radicals*

Prior Knowledge: Simplifying Radicals (multiplicative property of radicals)

Motivating the Topic: Products of radicals may simplify even when the original factors do not.

$\sqrt{AB} \cdot \sqrt{CB}$ (use any relatively prime A, B, and C)

Lesson Suggestions: If the above A, B, and C are chosen sufficiently large, students will complain about computing the combined radicand: $AB \cdot CB$

In general it is beneficial to avoid computing the combined radicand, as factored form is going to be necessary for simplifying our result.

Avoid combining radicands and instead factor the factors from the start:
$\sqrt{AB} \cdot \sqrt{CB} \to \sqrt{A}\sqrt{B}\sqrt{C}\sqrt{B} \to B\sqrt{AC}$

This lesson also includes products of binomials that involve square-roots. Specifically, we prepare students for the future task of rationalizing denominators by making sure they see products of conjugates.

Learning Outcomes: Students will multiply and simplify expressions involving square-roots.

Topic: *Rationalize Denominators: divide radical expressions*

Prior Knowledge: Division of complex numbers
Binomial conjugation (difference of squares)

Motivating the Topic: This topic used to be relevant when decimal approximations for square roots needed to be looked up in a table. Dividing by an integer is easier than dividing by an extended decimal approximation.

Have students recall how we handled division of complex numbers.

Lesson Suggestions: Students don't *have* to simplify the fraction or radicals first, but it helps.

Identify the "smallest" radical necessary in order to rationalize the denominator (in the monomial case).

Multiply the numerator and denominator by the appropriate conjugate (in the binomial case).

Learning Outcomes: Students will rewrite rational expressions in an equivalent form that does not use square roots in the denominator.

Topic: *Solving Radical Equations*

Prior Knowledge: Simplifying Radicals

Motivating the Topic: We know how to manipulate, simplify, and combine expressions involving square-roots.
Let's use what we know to solve equations that involve square-root expressions.

Lesson Suggestions: Principal strategy: isolate the single square-root term and square both sides.
Students are not expected to solve equations in which more than one square-root term appears.

It is important to remind students that their answers *must* be confirmed as solutions.
We should have already seen the idea of extraneous solutions back in "solving rational equations".

Extraneous solutions appear because "squaring both sides" of an equation has the potential to introduce equality where none existed before: $-3 \neq 3$ but $(-3)^2 = (3)^2$.

Learning Outcomes: Students will understand how extraneous solutions arise in the context of solving equations involving square-roots.
Students will solve equations involving square-roots and check their results to confirm their answers are solutions.

UNIT FIVE: TRIGONOMETRY

Topic: *Angle Measure & Similar Triangles*

Prior Knowledge: Many students have strong recollection regarding proportionality of similar shapes

Motivating the Topic: Measuring angles cannot be the same as measuring distance - because even with the same angle, distance is smaller near to the vertex of the angle, and distance grows as we consider distance measured farther away from the vertex.

Instead, angles are measured as *rotation* - with an easy unit of measure being a "full-rotation"

Because we often consider amounts of rotation that are much less than a full-rotation, it is useful to assign a large value (with a lot of factors, such as 360) to a full-rotation, so that we may split it up more easily.

Lesson Suggestions: If we assign a full-rotation to be 360 "degrees", then what is a half-rotation? A quarter-rotation?
This is useful discussion because we'll eventually be involving angles in radian measure.

Similar shapes have the same angles, but different "sizes". This is usually the manner in which students recall similar shapes.

If the ratio for one pair of corresponding sides is 1:2, then this ratio will be consistent for the other pairs of sides as well.
The "new" recognition here is that the ratios within a single triangle are constant, regardless of how the triangle is scaled.

So called "special" right triangles (next lesson) will reinforce this perspective, though it can be addressed now - with any triangle:
A triangle has side lengths a, b and c. A new triangle, similar to the first (by a scale factor of p - pick any value) has what side lengths? Draw pictures, label side lengths, ask students about the ratio of sides in each triangle - are they consistent? What if we had used a different scale factor?

Learning Outcomes: Students understand degrees as representing angle measurements in terms of fractions of a full-rotation.
Students understand the ratios between the sides of a triangle are unaffected by scale factor.

Topic: *"Special" (Right) Triangles*

Prior Knowledge: Similar triangles
Angle measurement in degrees

Motivating the Topic: Some triangles have computable ratios:

- Equilateral triangles are not right triangles, but all possible ratios of sides are very easy.
- Isosceles triangles have one pair of sides in an easy ratio.
- The isosceles right triangle allows us to use Pythagorean Theorem to determine the hypotenuse in terms of the congruent sides.
- Cutting the equilateral triangle in half creates a right triangle in which the sides may also be computed in terms of the initial equilateral side.

Lesson Suggestions: Isosceles right triangle sides are always in $1 : 1 : \sqrt{2}$ proportion.
30-60-90 triangles are always in $1 : \sqrt{3} : 2$ proportion.

It is **not** necessary that students address the separate ratios in either "special" triangle by their trigonometric names: sine, cosine, or tangent.
The trigonometric ratios will be named in the next topic...

Learning Outcomes: Students will apply the relevant proportions to "special" right triangles with one known side in order to determine the lengths of the remaining sides.

Topic: *Trigonometric Ratios*

Prior Knowledge: Similarity
Interior angles of a triangle sum to 180°

Motivating the Topic: In any right triangle, the two unspecified angles are acute and sum to 90°.
Knowing one of the acute angles means we know *all three* angles.
Having all three angles, all possible triangles are similar - meaning their side-ratios are constant (by what we learned from the two prior topics).
In summary: just a single acute angle establishes a fixed family of similar right triangles - all with the exact same side-ratios

Lesson Suggestions: Because one acute angle determines the ratios (for any right triangle drawn with the given acute angle), it is important to name the possible ratios relative to the given angle.
Begin by naming the *legs* relative to the given angle: "opposite" and "adjacent."
The hypotenuse is always immediately recognizable in a right triangle.
Based on sides O, A, and H - we can create 6 different ratios, and we can organize those 6 ratios into three different pairs of reciprocals.

mnemonic: SOH-CAH-TOA and then secant, cosecant, and cotangent as reciprocals of the three "main" ratios.

Knowing two sides of a right triangle also determines these ratios – because Pythagorean Theorem uniquely determines the third side (and knowing the sides means we know the ratios).

Learning Outcomes: Students understand the nature of trigonometric ratios as constant-valued across all sizes of a family of similar triangles.
Students compute the value of trigonometric ratios from a given pair of sides of a right triangle.
Given a specific value for a trigonometric ratio, students will construct a corresponding right triangle and infer the values for the other trigonometric ratios.

Topic: *Solving Right Triangles - Inverse Trig Functions*

Prior Knowledge: The unique relationship between an individual acute angle and the ratios created by a corresponding right triangle.

Motivating the Topic: If each acute angle determines a set of ratios (sine, cosine, tangent), does it work the same in reverse?
Can we determine a specific angle from a given ratio?
We have seen that a given ratio means that we can determine the *other ratios* - but what is the exact angle?)

Lesson Suggestions: It is a matter of personal taste, but the "inverse" notation ($\sin^{-1}(r)$) has the potential to mislead students about the nature of the inverse-trig functions. Students tend to misinterpret the "-1" as an exponent rather than as "inverse" - mainly due to the fact that the functional perspective is beyond the scope of this course and students lack the foundation for an introduction to inverse functions.
Instead, the "arc" notation ($\arcsin(r)$) suggests that we are computing an arc (or rotation) from our given ratio. This approach has the benefit of being unabiguous (with regard to notation) and it establishes the inverse functions as functions in their own right (rather than being defined by their relation to our standard trig functions strictly as inverses).

A bit of history can be useful here as well. In answering the motivating questions, we might consult a table of sine values. From the table, we can see that our desired angle is between two degree measurements. Technology has made these tables unnecessary, so rely on computer/calculator to find a sufficiently precise decimal approximation.

This is a good opportunity to address compounding error:
Given that $\tan(\theta) = \frac{a}{b}$ (choose some values), find an approximate value for θ
Use the approximated θ to compute $\cos(\theta)$ or $\sin(\theta)$
Use Pythagorean Theorem to find the exact value of the hypotenuse.
Use the length of the hypotenuse to compute the exact value for $\cos(\theta)$ or $\sin(\theta)$ and compare.

Learning Outcomes: Students will use technology to approximate the measure of angles that correspond to given trigonometric ratios.
Students do *not* need to understand the functional relationship between trigonometric functions and their inverses.

Topic: *Solving Right Triangles*

Prior Knowledge: The unique relationship between an individual acute angle and the ratios created by a corresponding right triangle.

Motivating the Topic: If an acute angle is enough to determine the specific family of similar right triangles, what else do we need to determine the *exact* triangle? (i.e. not just the ratios, but the length of all sides)

Lesson Suggestions: From the family of similar right triangles, we need to identify one specific triangle.
Knowing just one side length is enough, because we know the ratios of all sides to each other (via similarity).

A clinometer can be used to measure the angle of inclination to the top of any vertical object.
Knowing the distance from the clinometer to the base of the vertical object gives one sides of the right triangle - enough to determine (estimate) the height of the vertical object.
Alternately, knowing the height of the vertical object is enough to instead estimate the distance from the clinometer to the base of the object.

Learning Outcomes: Given an acute angle and a side length, students will determine the remaining two sides of the corresponding right triangle.

Topic: *Angle Measure : Radians*

Prior Knowledge: Measurement of angles in terms of degrees
Measurement of angles as fractions of full-rotations
Circumference of a circle

Motivating the Topic: $360°$ is a completely fabricated quantity in representing a full-rotation
Arc-length would be much more reasonable, as we think of rotation as circular...
But the arc-length for angles would change, depending on the radius we used in drawing the arc

Lesson Suggestions: The good news is that the ratio: $\frac{\text{arc-length}}{\text{radius}}$ is constant for all angles.
Circles (full-rotations) have circumference (arc-length) equivalent to $2\pi r$, but when we then divide by the radius, we get the constant 2π.

In radians, 2π represents a full-rotation (instead of $360°$ in degrees).
$180°$ represents a half-rotation, which would be π in radians (as half of 2π, just as 180 is half of 360).
This same idea applies to quarter-rotations, eighth-rotations, sixth-rotations, etc.

Rotations in the counter-clockwise direction are considered "positive" rotations.
Rotations in the clockwise direction are considered "negative" rotations.
Rotations are "co-terminal" when both their initial and terminal sides coincide. They "look identical" when drawn. The difference in their rotations is an integer multiple of 2π (or of $360°$ in degrees).

Learning Outcomes: Students will convert angles between degree and radian measures.
Students will distinguish between positive and negative angle measurements.
Students will identify co-terminal pairs of angles as well as formulate co-terminal angles from a given angle.

Topic: *Coordinate Plane Trigonometry*

Prior Knowledge: Right triangle definitions for sine, cosine and tangent
Angles represented as positive or negative (and co-terminal angles)

Motivating the Topic: Can we consider sine, cosine or tangent ratios for negative angles? Triangles do not "care" about negative measurements - but if we align our right triangles along the positive x-axis *and* allow our legs to sometimes have negative length - we can make sense of this question.

Lesson Suggestions: Start by placing a point in QI and determining the sine, cosine, and tangent (and reciprocals) for the angle formed by the x-axis and the line from the origin to your chosen point.

Avoid going to QII or QIII next, as it is easier for students to grasp the "reflection" of our right triangles from QI to QIV (particularly when it comes to our main restriction - keeping the right triangles drawn with one leg on the x-axis...)

Resolve having sine, cosine, and tangent of negative acute angles (by choosing a point in QIV), sine and tangent become negative, but cosine remains identical to QI.

Once we've established that our right triangles on the coordinate plane have legs on the x-axis (for QI and QIV), it is easier to argue that we maintain that construction in QII and QIII.

In this lesson, it is not necessary to address reference angles (we will do that with the unit circle - where we have "nice" angles and "nice" ratios). All that is necessary is to establish the positive and negative behavior of the sine, cosine, and tangent functions in each quadrant (and also to use their behaviors to identify which quadrant θ must come from).

It is also relevant to formally recognize the behavior of each trig function at the quadrantal angles.

Learning Outcomes: Given a point on the coordinate plane, students will determine the trigonometric ratios for the angle whose intial side is the x-axis and whose terminal side is the line connecting origin to the given point.

Given the value of a single trigonometric ratio for an undetermined angle, and the sign of a second trigonometric ratio (independent of the known ratio) for the same angle, students will determine the exact value for each of the five unspecified trigonometric ratios.

Topic: *Unit Circle*

Prior Knowledge: Trigonometric functions for right triangles (extending to angles on the coordinate plane)
"Special" triangles and their ratios
Radian measure and conversion between radians and degrees

Motivating the Topic: Instead of considering points on the coordinate plane (which cause us to start with the sides of our right triangle, without even needing the angle measurement), we consider starting with an angle measure.

An angle positioned with its vertex at the origin establishes a line (ray) as its terminal side - *there are so many points on that line!* Which one should we use?

We can standardize our choice (no matter which angle we start with) by always selecting the point that will give us a hypotenuse with length 1.
This choice also gives us a familiar shape on which our coordinate pairs will lie.

Lesson Suggestions: Our emphasis for this topic shifts from starting with side lengths to starting with angles.
Start with the "nicest" angles we can: $30°$, $45°$, and $60°$
Stick with our construction requirement that one leg *must* lie on the x-axis.
In each triangle we have one known side: Hypotenuse = 1
Determine the coordinate values from the ratios of our "special" triangles.
Before moving to the other quadrants, establish the **radian measure** for our "special" angles.
Also, before moving beyond QI, set the expectation that these first quadrant angles will be our "reference" angles for the reflected triangles in the other quadrants.

Now we're ready to break out of QI: Reflect into the second quadrant - emphasize that our θ is always measured from the *positive* x-axis, while our reflection is measured from the *negative* x-axis (so $\theta = \pi - \theta_r$ where θ_r is our first quadrant "reference angle").
Repeat for QIII and QIV

Learning Outcomes: Students will understand the symmetries present in the unit circle, and use them to determine the value of sine, cosine, and tangent for any "major" angle between -2π and 2π.

Topic: *Graphing Sine & Cosine*

Prior Knowledge: Behavior of sine and cosine in each quadrant
Unit Circle

Motivating the Topic: The Unit Circle can give students an odd impression of the graphs for sine and cosine. (Namely, that x is $\cos(\theta)$ and y is $\sin(\theta)$.)

It is important to distinguish between the unit circle and what we attempt to do in this lesson. (Switch to discussing $y = \sin(x)$ and $y = \cos(x)$ - which is especially difficult when students do not have a background in discussing *functions*!)

Lesson Suggestions: We do know the values of sine and cosine (as θ varies from 0 to 2π) from the unit circle.
Instead of going straight to the graph - begin by building a table of x and y values for $y = \sin(x)$ as x takes on "major" angles from 0 to 2π.
Plot the points as given by the table of values.
Do the same for cosine.

Note that both sine and cosine are restricted to y-coordinates between -1 and 1. (Which makes sense because our ratios for leg-to-hypotenuse cannot grow beyond those values!)

Also note the periodic nature of the graphs for sine and cosine. As we "continue to take trips around the unit circle" we will create a new oscillation for each successive round of co-terminal angles.
Discussing the effect of the amplitude parameter is pretty straightforward: multiplying the sine or cosine ratio by a value, A, means that our maximum and minimum will be affected.

The effect of the frequency parameter, B in $y = \sin(Bx)$, can be seen as "affecting the speed at which we travel around the unit circle" when generating oscillations. For example, $y = \sin(2x)$ means that we will generate oscillations at twice the "normal" rate. So we have 2 full oscillations between 0 and 2π.

Period is the width for a single oscillation. And if B determines how many oscillations we have between 0 and 2π, it makes sense that each has width $\frac{2\pi}{B}$.

Learning Outcomes: Students will graph oscillations of sine and cosine - making adjustments for changes in amplitude and frequency.

Topic: *Proving Trigonometric Tautologies*

Prior Knowledge: Pythagorean Theorem
Reciprocal relationships between trigonometric ratios

Motivating the Topic: This topic is an excercise in verifying the equivalence of (seemingly different) trigonometric expressions.

The primary example is that $\sin^2(\theta) + \cos^2(\theta) = 1$ - but why should we even believe that?

Lesson Suggestions: We already know the ratios for all six trigonometric functions in terms of "opposite" (O) and "adjacent" (A) legs, and the "hypotenuse" (H).
So $\sin^2(\theta) + \cos^2(\theta)$ can be rewritten as $\left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2$
Then squaring each fraction, we have: $\frac{O^2+A^2}{H^2}$
But then Pythagorean Theorem tells us that $O^2 + A^2$ is H^2
So this fraction $\frac{O^2+A^2}{H^2}$ is just $\frac{H^2}{H^2}$, which can only be 1!
We have now confirmed that indeed, $\sin^2(\theta) + \cos^2(\theta) = 1$

This same strategy can be used to "prove" the equivalence of any trigonometric tautology.

Strictly speaking, the "proof" process requires each side to be manipulated independently of the other. In other words, students should be discouraged from using additive or multiplicative equality properties to manipulate both sides of the given equation. Instead, each side should be manipulated until they reveal a literal equality.

Marta Hidegkuti has an excellent collection of examples and problems here.

Learning Outcomes: Students will apply their knowledge of trigonometric ratios to verify any given trigonometric tautology.

Topic: *Solving Trigonometric Equations*

Prior Knowledge: Unit Circle
Solving linear and quadratic equations

Motivating the Topic: We have seen the symmetries that appear in the unit circle, causing sine to repeat the same ratio for more than one angle.

What does this mean for solving equations that involve trigonometric functions?

Lesson Suggestions: Given a straightforward equation such as $\sin(\theta) = -\frac{\sqrt{2}}{2}$, the most fundamental approach is to look on the unit circle for angles with the specified sine ratio.

Of course, we want students to be more sophisticated, so they should be encouraged to apply their understanding of the quadrants as well as their understanding of reference angles to the task of determining possible values for θ.

Students should be eased into "tricker" equations:
$2\sin(\theta) = -\sqrt{2}$
$2\sin(\theta) + \sqrt{2} = 0$

Ultimately, students should end up solving factorable trigonometric equations.

Note: In this course, students are only expected to find solutions between 0 and 2π.

Learning Outcomes: Students will solve linear and quadratic equations involving trigonometric functions, finding all solutions between 0 and 2π.

Topic: *Law of Sines*

Prior Knowledge: Triangle congruency theorems: ASA and SSA (ambiguous)

Motivating the Topic: It is important for students to realize that this lesson pushes beyond the strictly right-triangle perspective of prior topics. Triangles should always have their vertices labeled - and students should be aware of the naming convention for sides of an arbitrary triangle "ABC" (side "a" is actually $\overline{BC}$, which is opposite from the vertex "A", and so on). The core concept of the ASA triangle congruency theorem is that if two triangles share a certain configuration of congruent angles and side, then the two triangles *must* be identical.

But this can only be true if the given angles and side somehow determine the unspecified angle and sides. Students will likely be quick to point out that the unspecified angle is easily determined - but what about the *sides* then?

Lesson Suggestions: Law of Sines enables us to compute the value of the missing sides in an "ASA" triangle.

It can be useful to point out that there are two "variants" of the Law of Sines, obtained by taking reciprocals:
$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$ and $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

If students are unclear as to how the Law of Sines applies when we do not have a right triangle, it can be illustrated by choosing any vertex and drawing an altitude. From the two right triangles that are created, we can easily derive one of the equalities in the Law of Sines.
Students should also see the attempted application of Law of Sines to a SSA triangle.
When finding the first unspecified angle, students should recognize that we are solving a trigonometric equation for sine (equal to a positive ratio) which will have solutions in QI and QII.
Students should also recall that there is no SSA triangle congruency theorem - and this is precisely why. There can be more than one possible triangle with the given configuration of sides and angle.

Learning Outcomes: Students will apply the Law of Sines to find the unspecified angle and sides from a given ASA triangle.
Students will recognize the difference between an ASA and SSA configuration, and they will understand the potential ambiguity that arises from applying the Law of Sines to the SSA situation.

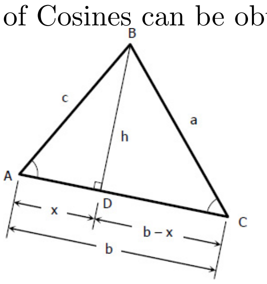
Topic: *Law of Cosines*

Prior Knowledge: Triangle congruency theorems: SSS and SAS

Motivating the Topic: If Law of Sines was covered first, then students already understand that we are tasked with "solving" non-right-triangles.

There are two triangle congruency theorems that Law of Sines did not address - SSS and SAS. But, as before, these configurations of known sides (and angle) must somehow determine the unknown angles (and side).

Lesson Suggestions: Law of Cosines can also be derived in a similar fashion to Law of Sines. Select a vertex and draw an altitude - from the two right triangles, the Law of Cosines can be obtained.



For example:
$a^2 = h^2 + (b - x)^2$ from Pythagorean Theorem
$a^2 = h^2 + b^2 - 2bx + x^2$ square binomial
$a^2 = h^2 + x^2 + b^2 - 2bx$ rearrange terms
$a^2 = c^2 + b^2 - 2bc\cos(A)$ because $h^2 + x^2$ is c^2
and because $\cos(A) = \frac{x}{c}$

It is useful for students to understand the common structure of the three variants for the Law of Cosines.

Note: It should be strongly suggested that students avoid performing calculations with approximated values, opting instead to use only those measurements that can be exactly determined from the given information.
Compounding error is a serious issue in applications of trigonometry (indeed, all applied mathematics) and students should be made aware of it early in their studies. For example, rounded values for the interior angles of a triangle do not have to sum up to 360°.
For example: $59.\overline{6}, 59.\overline{6}, 60.\overline{6}$ sum to 180 unless they are rounded off - this example creates a discripancy for any chosen level of error tolerance.
For Law of Sines or Cosines, the "sum to 180°" property should only be used to check answers.

Learning Outcomes: Students will apply the Law of Cosines to determine unspecified angles (and side) from SSS or SAS triangles.
Students will understand the nature of compounding error and they will attend to the precision of their decimal approximations.

UNIT SIX: EXPONENTIALS AND LOGARITHMS

Topic: *Exponential "Functions"*

Prior Knowledge: Integer and Rational exponents
Students largely do not have the foundation necessary to understand this topic from a functional perspective

Motivating the Topic: In a polynomial expression, we will see x raised to a power - but what happens if the roles are reversed?
What if we instead have a *constant* base with x in the *exponent*?

For what kind(s) of situation would this be useful?
Any situation that has a fixed (multiplicative) rate of growth.

Hypothetical situtations such as the wheat-and-chessboard problem or asking students about starting with a penny and doubling every day for a month.

Lesson Suggestions: Discuss situations where a fixed rate of growth (or decay) occurs. Population growth, fixed rate investements, radioactive half-life, etc. This is *not* the place to discuss these topics in depth, but it is useful to make students aware of why we might use a variable exponent (to represent the number of times to multiply the fixed growth rate).

Our initial focus is to review exponents: positive and negative; integer and rational.
Students should be asked to solve basic exponential equations such as $9^x = 81$, $9^x = \frac{1}{81}$, and $9^x = 3$.

Students should recognize that the base, B, cannot be negative (nor zero, nor 1) without causing issues. If we use a variable as an exponent, then we are suggesting that the exponent could represent *any* value - and a negative base would not "work" with exponents such as $\frac{1}{2}$ or $\frac{3}{4}$.

Learning Outcomes: Students will evaluate exponential expressions by substituting a specified value for the variable in the exponent.
Without a calculator, students will solve basic exponential functions of the form $B^x = R$ where R is an integer or rational power of B.

Topic: *Logarithmic "Functions"*

Prior Knowledge: Exponential "Functions"
Integer and Rational exponents
Students largely do not have the foundation necessary to understand this topic from a functional (or inverse) perspective

Motivating the Topic: Solving equations of the form $B^x = R$ is easy when R is a "nice" power of B. But what about when that's not the case? e.g. $3^x = 15$

We have been in this situation before, when solving $x^2 = R$. It was easy when R was a "perfect square", but what did we do when facing $x^2 = 15$? We may have estimated a value between 3 and 4, but the actual value for x is an irrational (infinite, non-repeating decimal) number. We couldn't possibly write down its decimal representation. Instead, we came up with a symbolic way to represent "the number whose square is 15": $\sqrt{15}$.

In this case we do the same thing. We have an irrational value for "the power of 3 whose result is 15" and we need a finite way to write down its *exact* value.

Lesson Suggestions: Begin by connecting the number we seek to represent: "the power of 3 whose result is 15" to the symbolic notation we use in representing it: $\log_3(15)$

Instead of continuing with $\log_3(15)$, switch to an easier example: $3^x = 81$
Then x is "the power of 3 whose result is 81"
Symbolically, x is $\log_3(81)$, and we recognize that $\log_3(81)$ is the same as 3.

This pattern extends to any "basic" exponential equation:
$B^x = R$ is the same as saying "x is the power of B whose result is R."
This sentence translates to $x = \log_B(R)$.

Now students should work on evaluating logarithmic expressions by converting them to exponential equations:
$\log_{16}(8)$ is what we want (call it "x")
so $x = \log_{16}(8)$, which converts to $16^x = 8$
Find a *common base* to force equality of exponents
$\left(2^4\right)^x = 2^3$ so then $2^{4x} = 2^3$ and therefore $x = \frac{3}{4}$

Learning Outcomes: Students understand the meaning of logarithmic notation - $\log_B(R)$ refers to "the power of B whose result is R."
Students will convert between exponential and logarithmic equations.
Students will evaluate logarithmic expressions without a calculator.
Students do *not* need to understand the "inverse" relationship of logarithms.

Topic: *Logarithmic Properties*

Prior Knowledge: Exponential properties

Motivating the Topic: We can simplify exponential expressions such as $B^x \cdot B^y$ or $(B^x)^y$

Because logarithms *are* exponents, they will have analogous properties

Lesson Suggestions: How might we express the exponential property: $B^x \cdot B^y = B^{(x+y)}$?
The exponential version expresses an equality of the results in terms of the exponents - for logarithms, we will express an equality of exponents in terms of the results. So it will help to define: $M = B^x$ and $N = B^y$.
Now the original can be written: $M \cdot N = B^{(x+y)}$
Which can be converted to logarithms: $\log_B(M \cdot N) = x + y$
And now we note what x and y are: $x = \log_B(M)$ and $y = \log_B(N)$
(from the definition of M and N)
So: $\log_B(M \cdot N) = \log_B(M) + \log_B(N)$

Use the same notation to convert $\frac{M}{N} = B^{(x-y)}$ and $(B^x)^y$.

Don't forget to also convert exponent properties such as $B^0 = 1$ and $B^1 = B$.

Students should "expand" and "condense" logarithmic expressions - but don't rely solely on algebraic expressions for this! e.g. $\log_5(25\sqrt{5})$ as well as $\log_B(x^2\sqrt{y})$

Learning Outcomes: Students will apply the logarithmic properties to expand or condense both algebraic (as well as numeric) logarithmic expressions.

Topic: *Compound Interest*

Prior Knowledge: Evaluating exponential expressions

Motivating the Topic: This is not a lesson on continuously compounding interest

Lesson Suggestions: $P =$ principal (the initial amount invested)
$t =$ time
$r =$ interest rate (per unit of t)
$n =$ number of times to compound interest (per unit of t)
$A =$ final amount (after t units of time have passed)

Then $A = P\left(1 + \frac{r}{n}\right)^{(n \cdot t)}$
$n \cdot t$ is the number of times we *compound* (monthly for 5 years would be 60 times compounded).
$\frac{r}{n}$ is the interest earned at each compounding (i.e. when compounding monthly, we would not earn the *annual* interest for each *month*!)

Have students compute the value of an investment for a fixed time and fixed n, with different rates.
Have students compute the value of an investment for a fixed time and fixed rate, with different n.
Preciocious students can be asked to compound daily, hourly, minute-ly, second-ly, etc...
An estimate of the behavior for very large n can be reserved for the calculator lesson - in order to approximate a value for e (as a look ahead to precalculus).

Learning Outcomes: Students will apply the discretely compounded interest formula to compute the outcome of an investment from the given parameters.
Students will understand (and explain) the impact that increasing or decreasing n has on the outcome of an investment.

Topic: *Solving Exponential Equations (no calculator)*

Prior Knowledge: Evaluating exponential expressions

Motivating the Topic: Exponents allow us to express the same value in multiple ways (in a manner similar to fractions: $\frac{1}{2} = \frac{3}{6}$).
For example: $81^{\frac{1}{2}} = 3^2$
As with fractions, where a common denominator forced equal fractions to have equal numerators, equivalent exponential expressions, written with a common base, will have equal exponents.

Lesson Suggestions: Complete the example: 81 is 3^4 and so $\left(3^4\right)^{\frac{1}{2}}$ will clearly have the same exponent as 3^2.

Students should be asked to solve equations with linear expressions taking the place of the exponent:
e.g. $B^{(mx+n)} = B^C$
extending to equations where the bases appearing on each side require conversion to a common base.

Students do not need to solve exponential equations with coefficients.

Learning Outcomes: Students will solve exponential equations with different (yet compatible) bases and exponents that are linear forms in the specified variable.

Topic: *Solving Exponential Equations (with calculator)*

Prior Knowledge: Converting between logarithmic and exponential equations
Compound interest

Motivating the Topic: We established logarithms in order to (quickly) write down values of the form "the power of B whose result is R."
So we know that the solution to $3^x = 15$ is $x = \log_3(15)$... but *what number is that?!*

As with square roots (e.g. $\sqrt{15}$ is between 3 and 4, closer to 4), we can estimate the value of a logarithm.
$x = \log_3(15)$ is between 2 ($\log_3(9)$) and 3 ($\log_3(27)$).
But, also as with square roots, we can (hopefully) rely on technology to do better.

Lesson Suggestions: We are limited because our calculators (most of them) do **not** allow us to specify a base!
We only have one button: "log"
(which students should be informed is $\log_{10}()$)
So, does this mean we always have to write our equations with base 10?

Yes. And no.
Go back to the original equation (or convert back to exponential form from $\log_3(15)$): $3^x = 15$
We know (?) that $10^{\log_{10}(3)}$ will be 3;
(ten raised to "the power of ten that results in 3" *must* give 3 as a result!)
and $10^{\log_{10}(15)}$ will be 15.
So: $\left(10^{\log_{10}(3)}\right)^x = 10^{\log_{10}(15)}$
And therefore: $x \cdot \log_{10}(3) = \log_{10}(15)$
So using our calculator for $\frac{\log_{10}(15)}{\log_{10}(3)}$ will give us a decimal approximation for $\log_3(15)$!

Apply the same strategy to $x = \log_B(R)$, illustrating the "change of base" formula for logarithms:
$\log_B(R) = \frac{\log_{10}(R)}{\log_{10}(B)}$

It should be pointed out to students that "ln" (on the calculator) is the "natural logarithm" and it refers to $\log_e()$, where $e \approx 2.71828...$
"e" will become more relevant in precalculus. For now, it is (more or less) just another number like π...

Learning Outcomes: Students will apply the change of base formula to approximate logarithms.
Students will solve exponential equations that have irrational solutions.