MAT.1180 - Mathematical Concepts and Applications

Chapter 5 (Aug, 27)

- Number Theory: **Prime** and **Composite Numbers**.
  - The set of **Natural numbers**, aka, **Counting numbers**, denoted by $\mathbb{N}$, is

    $$\mathbb{N} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6, ...\}$$

  - If a and b are natural numbers, a is divisible by b if operation of dividing a by b leaves a remainder of 0. This is the same of saying that b is a **divisor** of a, or b **divides** a. All three statements are symbolized by writing

    $$b|a$$

  - Divisibility
    * 2 - The last digit is 0, 2, 4, 6, or 8.
    * 3 - The sum of all digits is divisible by 3.
    * 4 - The last 2 digits form a number divisible by 4.
    * 5 - The last digit is 0, or 5.
    * 6 - The number is divisible by both 2 and 3.
    * 8 - The last 3 digits form a number divisible by 8.
    * 9 - The sum of all digits is divisible by 9.
    * 10 - The last digit is 0.
    * 12 - The number is divisible by both 3 and 4.
  - A **Prime Number** is a natural number greater than 1 that has only itself and 1 as factors.
    * 1 is NOT prime.
    * The smallest prime is 2.
  - A **Composite Number** is a natural number greater than 1 that is divisible by a number other than itself and 1.
    * The smallest composite number is 4.
    * Any natural even number bigger than 2 is composite.
  - The natural numbers can be partitioned into three groups: 1, primes, and composites.
  - **The Fundamental Theorem of Arithmetic**
    * Every composite number can be uniquely expressed as a product of prime numbers.
  - [**ex**] Find the prime factorization of the following numbers

    $$30,\ 64,\ 700$$

  - [**ex**] Find the **Greatest Common Divisor (GCD)** of the following groups of numbers.
    1. 216 and 234
    2. 225 and 825
  - [**ex**] Find the **Least Common Multiple (LCM)** of the following groups of numbers.
    1. 144 and 300
    2. 18 and 30
  - [**ex**] A movie theater runs two documentary films continuously. One runs for 40 minutes and a second runs for 60 minutes. Both movies begin at 3 : 00 PM. When will the movies begin again at the same time?
- The Integers; Order of Operation

- The **Whole Numbers** is the set consisting of 0 and the natural numbers, i.e.,

$$\text{Whole Numbers} = \{0,\ 1,\ 2,\ 3,\ 4, ...\}$$

- The **Integers**, $\mathbb{Z}$, is the set consisting of whole numbers and the negative of natural numbers,

$$\mathbb{Z} = \{...,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4, ...\}$$

  In particular,
  * The negative of natural numbers is called **Negative Integers**, and denoted by $\mathbb{Z}^-$.
  * The natural numbers is also called **Positive Integers**. We can use notation $\mathbb{Z}^+$, but mostly people just use $\mathbb{N}$.
- [**ex**] Write the following numbers on the number line

$$5,\ -3,\ 0,\ -7,\ 6$$

- [**ex**] Use "<" or ">" to connect the following groups of numbers.
  1. 8 and -7
  2. 4 and -6
  3. -3 and -9
  4. 0 and -2
- The **Absolute Value** of an integer a, denoted by $|a|$, is the distance from 0 to a on the number line.
  * Because absolute value describe a distance, it is never negative.
  * [**ex**] Evaluate the followings:

$$|8|,\ |-5|,\ |0|,\ -|-3|,\ -|4|$$

- [**ex**] Evaluate.
  1. $-2 + (-8)$
  2. $-3 + 5$
  3. $-8 - 4$
  4. $10 + (-2)$
  5. $12 - (-8)$
  6. $-6 - (-4)$
  7. $-3 - 7$
  8. $3(-5)$
  9. $(-4)(-3)$
  10. $-8(0)$
  11. $-2(-3)(-5)$
  12. $-2(3)(-5)$
  13. $(-3)^2$
  14. $-(3)^2$
  15. $(-2)^3$
  16. $(-2)^4$
  17. $-20 \div 4$
  18. $12 \div (-3)$
- [**ex**] Evaluate.
  1. $3 + 7 \cdot 2$
  2. $8 \div 4 \cdot 2$

3. $8 \div (4 \cdot 2)$
4. $3^2 - 24 \div 2^2 \cdot 3 + 1$
5. $(-4)^2 - (10 - 13)^2(-2)$

- The **Rational Numbers**
  - The set of **Rational Numbers**, denoted by $\mathbb{Q}$, is the set of all numbers which can be expressed in the form $\frac{a}{b}$, where a and b are integers and b is not 0. The integer a is called the **Numerator**, and b is called the **Denominator**.
    * If positive integers a and b shares no common factor other than 1, a and b are said to be **Relatively Prime**.
    * If $\frac{a}{b}$ is a rational number and c is any number other than 0, then

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

    * $\frac{a}{b}$ is said to be in **reduced form** if a and b are integers sharing no common factor other than ± 1.
    * [**ex**] Reduce/simplify the following numbers.

$$\frac{12}{4}, \ \frac{20}{12}, \ \frac{-18}{30}$$

    * [**ex**] Convert the following **Mixed Numbers** to **Improper Fractions**.

$$3\frac{3}{7}, \ -5\frac{1}{3}$$

    * [**ex**] Convert the following improper fractions to mixed.

$$\frac{15}{4}, \ -\frac{23}{5}$$

    * [**ex**] Express each rational as a decimal.

$$\frac{5}{8}, \ \frac{8}{11}$$

    * [**ex**] Express the following decimals as reduced fractions.

$$0.7, \ 0.2, \ 0.26, \ 0.048, \ 0.\overline{6}, \ 0.\overline{36}$$

    * [**ex**] Determine if 1 or $0.\overline{9}$ is greater.
  - For fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$$

    * [**ex**] Evaluate. The final answer must be simplified.
      1. $\frac{3}{8} \cdot \frac{5}{11}$
      2. $-\frac{6}{7} \cdot \frac{14}{3}$
      3. $3\frac{2}{3} \cdot 1\frac{1}{4}$
      4. $\frac{9}{11} \div \frac{5}{4}$
      5. $-\frac{8}{15} \div \frac{2}{5}$

6. $3\frac{3}{8} \div 2\frac{1}{4}$
7. $\frac{3}{4} + \frac{1}{6}$
8. $\frac{1}{5} + \frac{3}{4}$
9. $\frac{3}{10} - \frac{7}{12}$
10. $(\frac{1}{2})^3 - (\frac{1}{2} - \frac{3}{4})^2(-4)$
11. $(-\frac{1}{2})^2 - (\frac{7}{10} - \frac{8}{15})^2(-18)$

* **[ex]** Find the number halfway between $\frac{1}{3}$ and $\frac{1}{2}$.

- **Irrational Numbers**
  - The set of **Irrational Numbers** is the set of numbers whose decimal representations are neither terminated nor repeating.
    * The two most famous irrational numbers for now is π and e.
  - The **Principle Square Root** of a nonnegative number n, written $\sqrt{n}$, is the positive number that when multiplied by itself gives n.
  - Rules on Square Roots
    * If $a > 0$ and $b > 0$, then
    $$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \text{ and } \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$
    * If $a > 0$ and $b > 0$, then
    $$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$
    * $a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$
  - Evaluate. The final answer must be simplified or rationalized when applicable.
    1. $\sqrt{12}$
    2. $\sqrt{60}$
    3. $\frac{\sqrt{80}}{\sqrt{5}}$
    4. $\sqrt{2} + \sqrt{8}$
    5. $4\sqrt{8} - 7\sqrt{18}$
    6. $\sqrt{12} + \sqrt{18}$
    7. $\frac{2}{\sqrt{3}}$
    8. $\sqrt{\frac{2}{7}}$

- **Real Numbers and Their Properties**
  - Number system. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^c$, $\mathbb{Z}$, $\mathbb{N}$ and whole numbers.
  - **[Optional]** Properties of $\mathbb{R}$.
    * Closed under Addition and Multiplication.
    * Commutative under Addition and Multiplication.
    * Associative under Addition and Multiplication.
    * Distributive of Multiplication over Addition.
    * Additive Identity: 0.
    * Additive Inverse, aka, Opposite.
    * Multiplicative Identity: 1.
    * Multiplicative Inverse (except for 0), aka, Reciprocal.
  - **[Optional]** **[ex]** Identify the properties of the operation.
    1. $3 \cdot \sqrt{2} = \sqrt{2} \cdot 3$
    2. $(2 + 3) + 5 = 2 + (3 + 5)$

3. $(4 - \sqrt{3}) \cdot 5 = 20 - 5\sqrt{3}$
4. $1.5 + 0 = 1.5$
5. $2 \cdot \frac{1}{2} = 1$
6. $1 \cdot \sqrt{5} = \sqrt{5}$
7. $(-2) + 2 = 0$

- [**Optional**] [**ex**] Give an example to show that
  1. Irrationals are NOT closed under multiplication.
  2. Naturals are NOT closed under division.

- Properties of Exponents
  - Properties of **Exponents**
    * $b^m \cdot b^n = b^{m+n}$
    * $(b^m)^n = b^{mn}$
    * $\dfrac{b^m}{b^n} = b^{m-n}$
    * $b^0 = 1$ for $b \neq 0$
    * 0^0 is undefined.
    * $b^{-m} = \dfrac{1}{b^m}$
  - [**ex**] Simplify
    1. 7^0
    2. $4^{-2} \cdot 4^4$
    3. $\frac{2^6}{2^3}$
    4. $\frac{3^{12}}{3^{-5}}$
    5. $\frac{4^{-5}}{4^7}$
    6. $\frac{5^{-3}}{5^{-8}}$
  - A positive number is written in **Scientific Notation** when it is expressed in the form

$$a \times 10^n$$

    where $1 \leq a < 10$ and n is an integer.
  - [**ex**] Write each number in decimal notation
    1. 3.04×10^5
    2. 2.156×10^{-6}
  - [**ex**] Write each number in scientific notation
    1. 637000000
    2. 0.0000236
  - [**ex**] Evaluate. Leave the final answer in scientific notation.
    1. $(3.4 \times 10^5)(2.1 \times 10^{21})$
    2. $30120000 \times 0.0000057$