The Variance and standard deviation of a random variable

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April 16, 2017

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  integrating over the whole real line
- In either case, the standard deviation is the square root of the variance: it is denoted by σ_X
  $\sigma_X = \sqrt{Var(X)} = \sqrt{\sigma_X^2}$

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- Then $x - \mu_X$ is the distance the value x is away from the center, except that this distance is negative if x is less than the mean.
- $x - \mu_X$ is called the deviation from the mean.
- The variance is formed by squaring all the deviations for all the possible values of X, and then finding the expected value of the squared deviations.
- It is a kind of average distance away from the mean. It measures the amount of "spread" of the possible values of the RV: are they close to the mean in general, or more spread out?

Example: computing the variance for a finite RV (the one of Ex. 4.1)

- The random variable X = the number of TV sets with white cords in the shipment
  The possible values of X are 0, 1, 2
  $p(0) = \frac{6}{11}$, $p(1) = \frac{9}{22}$, $p(2) = \frac{1}{22}$
  We previously computed $\mu_X = \frac{1}{2}$
  We will now compute the three deviations from the mean and square them:

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- Now multiply each squared deviation by the probability for that value of x and add them together:
- $\sigma^2(X) = \frac{1}{4}\left(\frac{6}{11}\right) + \frac{1}{4}\left(\frac{9}{22}\right) + \frac{9}{4}\left(\frac{1}{22}\right)$

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- Now multiply each squared deviation by the probability for that value of x and add them together:
- $\sigma^2(X) = \frac{1}{4}\left(\frac{6}{11}\right) + \frac{1}{4}\left(\frac{9}{22}\right) + \frac{9}{4}\left(\frac{1}{22}\right)$
- $= \frac{6}{44} + \frac{9}{88} + \frac{9}{88} = \frac{30}{88} = \frac{15}{44}$

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- X has possible values 1, 2, 3, 4, 5, 6;
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- $= \frac{21}{6}$
- $= \frac{7}{2}$

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- $= (1 + 2 + 3 + 4 + 5 + 6)\left(\frac{1}{6}\right)$
- $= \frac{21}{6}$
- $= \frac{7}{2}$
- No accident, this is exactly halfway between 1 and 6, because this RV is uniformly distributed.

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- X has possible values 1, 2, 3, 4, 5, 6; each one has probability $\frac{1}{6}$
- $\mu_X = \frac{7}{2}$, now to compute the variance:

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- $Var(X) = \sum_{x=1}^{6} \left(x - \frac{7}{2}\right)^2 \cdot p(x)$

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- $Var(X) = \sum_{x=1}^{6} \left(x - \frac{7}{2}\right)^2 \cdot p(x)$
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- $= \left(-\frac{5}{2}\right)^2 \left(\frac{1}{6}\right) + \left(-\frac{3}{2}\right)^2 \left(\frac{1}{6}\right) + \left(-\frac{1}{2}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{3}{2}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{2}\right)^2 \left(\frac{1}{6}\right)$

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- $= \left(\frac{1}{6}\right) \left(\frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4}\right)$

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- $= \left(\frac{1}{6}\right) \left(\frac{70}{4}\right)$

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- $= \left(\frac{1}{6}\right) \left(\frac{70}{4}\right)$
- $= \frac{35}{12}$

A useful theorem about the variance

Theorem (The “computational formula” for the variance)

$\sigma_X^2 = E\left(X^2\right) - \mu_X^2$

This theorem provides an alternate way to compute the variance, which is often much faster than using the definition.

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- $= \left(\frac{1}{6}\right)(1 + 4 + 9 + 16 + 25 + 36)$

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- $= \left(\frac{1}{6}\right)(1 + 4 + 9 + 16 + 25 + 36)$
- $= \frac{91}{6}$

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- $= \left(\frac{1}{6}\right)(1 + 4 + 9 + 16 + 25 + 36)$
- $= \frac{91}{6}$
- Then compute $\sigma_X^2 = E(X^2) - \mu^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182}{12} - \frac{147}{12} = \frac{35}{12}$