The Expected Value of a random variable

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  summing over all possible values of X
  Here I am using the shorthand $p(x)$ to stand for $p(X = x)$
- For a continuous RV,
  $E(X) = \mu_X := \int x \cdot f(x)\mathrm{d}x$
  integrating over the whole real line

Example: computing the expected value for a finite RV (Ex. 4.1)

- A lot of 12 television sets includes 2 with white cords. If 3 of the sets are chosen at random for shipment to a hotel, how many sets with white cords can the shipper expect to send to the hotel?

- Examples from Miller, Irwin. John E. Freund's Mathematical Statistics with Applications, 8th Edition. Pearson, 20151203. VitalBook file.

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- $p(X = 2) = p(2) = \frac{\binom{2}{2}\binom{10}{1}}{\binom{12}{3}} = \frac{10}{220} = \frac{1}{22}$

Example: computing the expected value for a finite RV (Ex. 4.1, continued)

- So we have X with possible values 0, 1, 2, $p(0) = \frac{6}{11}$, $p(1) = \frac{9}{22}$, $p(2) = \frac{1}{22}$

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- So we have X with possible values 0, 1, 2, $p(0) = \frac{6}{11}$, $p(1) = \frac{9}{22}$, $p(2) = \frac{1}{22}$
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Example: computing the expected value for a continuous RV (Ex. 4.2)

- Certain coded measurements of the pitch diameter of threads of a fitting have the probability density

$$f(x) = \frac{4}{\pi\left(1+x^2\right)} \text{ for } 0 < x < 1$$

We will verify that this is a probability density (not done in the textbook) and then compute the expected value of the pitch diameter.

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- $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_0^1 \frac{4}{\pi(1+x^2)}\mathrm{d}x = \frac{4}{\pi}\int_0^1 \frac{1}{1+x^2}\mathrm{d}x$

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- $$f(x) = \frac{4}{\pi\,(1+x^2)} \text{ for } 0 < x < 1$$

  Compute the expected value of the pitch diameter.

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- Do this by substitution using $u = 1 + x^2$, so $\mathrm{d}u = 2x\mathrm{d}x$, $x\mathrm{d}x = \frac{\mathrm{d}u}{2}$

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- $= \frac{2}{\pi}\left[\ln(u)\right]_{x=0}^{x=1} = \frac{2}{\pi}\left[\ln(1+x^2)\right]_{x=0}^{x=1}$

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- $= \frac{2}{\pi} \left[\ln(u)\right]_{x=0}^{x=1} = \frac{2}{\pi} \left[\ln(1+x^2)\right]_{x=0}^{x=1}$
- $= \frac{2}{\pi} \left[\ln(2) - \ln(1)\right] = \frac{2}{\pi}(\ln(2)) = \frac{\ln 4}{\pi} \approx 0.44$