Conditional Probability and Independence

Comment: In the Bayesian point of view, probability is defined based on our "degree of ignorance" of the outcomes. In defining conditional probabilities we are in the situation of having decreased our ignorance somewhat, and we look at the effect that has on the probabilities.

Definition and notation: The **conditional probability** of A given B is the probability of A with the sample space restricted to B. (We know that B holds or has happened.) We only count the part of A that lies in B.

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

(Note: the sign := means that this is a definition of the quantity on the left.)

There is another notation used for the conditional probability of A given B:

$$P_B(A)$$

In either case you read it as "the probability of A given B".

In an **equally likely outcomes model**, we may equivalently define this probability as follows:

$$P(A|B) = \frac{\text{the number of outcomes in } A \cap B}{\text{the number of outcomes in } B} = \frac{|A \cap B|}{|B|}$$

Example: for the experiment "toss two balanced coins", am equally likely outcomes model is $S = \{HH, HT, TH, TT\}$ with the probability of each outcome being $\frac{1}{4}$. Then:

- The probability of two heads showing is $P(HH) = \frac{1}{4}$ (not a conditional probability)
- The probability of two heads showing, given that at least one of the coins shows a head, is computed based on the restricted sample space
  $B =$ "at least one coin shows a head" $= \{HH, HT, TH\}$
  Then, setting $A = \{HH\}$, we have $A \cap B = \{HH\}$
  So $P(A|B) = \frac{1}{3}$

Sidebar: Notice in this example that the "decreased ignorance" regarding the number of heads that show has increased the probability of two heads showing, from $\frac{1}{4}$ to $\frac{1}{3}$. Reducing ignorance will not always increase the original probability; it may decrease or not change at all. Look for examples where each of those things happens. (One is below.)

Observe that for events A and B which are mutually exclusive, $P(A|B) = 0$. (Why?)

Independent Events: The events A and B are **independent** if $P(A|B) = P(A)$.
In other words, two events are independent if the probability of one of them happening does not depend on whether or not the other one occurs.

Also, if $P(A|B) = P(A)$ then it is also true that $P(B|A) = P(B)$ so it makes sense to say they are independent of each other.

If A and B are **independent**, we can prove that also

$$P(A \cap B) = P(A)P(B) \text{ (Multiplication Law for Independent Events)}$$

. This is sometimes the easiest way to check to see if two events are independent.