A third option is to state a mathematical formula that the sample outcomes must satisfy.

A computer programmer is running a subroutine that solves a general quadratic equation, $ax^2 + bx + c = 0$. Her "experiment" consists of choosing values for the three coefficients a, b, and c. Define (1) S and (2) the event A: Equation has two equal roots.

First, we must determine the sample space. Since presumably no combinations of finite a, b, and c are inadmissible, we can characterize S by writing a series of inequalities:

$$S = \{(a, b, c) : -\infty < a < \infty, -\infty < b < \infty, -\infty < c < \infty\}$$

Defining A requires the well-known result from algebra that a quadratic equation has equal roots if and only if its discriminant, $b^2 - 4ac$, vanishes. Membership in A, then, is contingent on a, b, and c satisfying an equation:

$$A = \{(a, b, c) : b^2 - 4ac = 0\}$$ ■

Questions

2.2.1. A graduating engineer has signed up for three job interviews. She intends to categorize each one as being either a "success" or a "failure" depending on whether it leads to a plant trip. Write out the appropriate sample space. What outcomes are in the event A: Second success occurs on third interview? In B: First success never occurs? (*Hint*: Notice the similarity between this situation and the coin-tossing experiment described in Example 2.2.1.)

2.2.2. Three dice are tossed, one red, one blue, and one green. What outcomes make up the event A that the sum of the three faces showing equals 5?

2.2.3. An urn contains six chips numbered 1 through 6. Three are drawn out. What outcomes are in the event "Second smallest chip is a 3"? Assume that the order of the chips is irrelevant.

2.2.4. Suppose that two cards are dealt from a standard 52-card poker deck. Let A be the event that the sum of the two cards is 8 (assume that aces have a numerical value of 1). How many outcomes are in A?

2.2.5. In the lingo of craps-shooters (where two dice are tossed and the underlying sample space is the matrix pictured in Figure 2.2.1) is the phrase "making a hard eight." What might that mean?

2.2.6. A poker deck consists of fifty-two cards, representing thirteen denominations (2 through ace) and four suits (diamonds, hearts, clubs, and spades). A five-card hand is called a *flush* if all five cards are in the same suit but not all five denominations are consecutive. Pictured in the next column is a flush in hearts. Let N be the set of five cards in hearts that are *not* flushes. How many outcomes are in N? [*Note*: In poker, the denominations (A, 2, 3, 4, 5) are considered to be consecutive (in addition to sequences such as (8, 9, 10, J, Q)).]

		Denominations												
		2	3	4	5	6	7	8	9	10	J	Q	K	A
	D													
Suits	H	X	X				X				X	X		
	C													
	S													

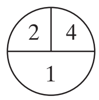
2.2.7. Let P be the set of right triangles with a 5″ hypotenuse and whose height and length are a and b, respectively. Characterize the outcomes in P.

2.2.8. Suppose a baseball player steps to the plate with the intention of trying to "coax" a base on balls by never swinging at a pitch. The umpire, of course, will necessarily call each pitch either a ball (B) or a strike (S). What outcomes make up the event A, that a batter walks on the sixth pitch? (*Note*: A batter "walks" if the fourth ball is called before the third strike.)

2.2.9. A telemarketer is planning to set up a phone bank to bilk widows with a Ponzi scheme. His past experience (prior to his most recent incarceration) suggests that each phone will be in use half the time. For a given phone at a given time, let 0 indicate that the phone is available and let 1 indicate that a caller is on the line. Suppose that the telemarketer's "bank" is comprised of four telephones.

(a) Write out the outcomes in the sample space.
(b) What outcomes would make up the event that exactly two phones are being used?
(c) Suppose the telemarketer had k phones. How many outcomes would allow for the possibility that at most one more call could be received? (*Hint*: How many lines would have to be busy?)

2.2.10. Two darts are thrown at the following target:



(a) Let (u, v) denote the outcome that the first dart lands in region u and the second dart, in region v. List the sample space of (u, v)'s.
(b) List the outcomes in the sample space of *sums*, $u + v$.

2.2.11. A woman has her purse snatched by two teenagers. She is subsequently shown a police lineup consisting of five suspects, including the two perpetrators. What is the sample space associated with the experiment "Woman picks two suspects out of lineup"? Which outcomes are in the event A: She makes at least one incorrect identification?

2.2.12. Consider the experiment of choosing coefficients for the quadratic equation $ax^2 + bx + c = 0$. Characterize the values of a, b, and c associated with the event A: Equation has complex roots.

2.2.13. In the game of craps, the person rolling the dice (the *shooter*) wins outright if his first toss is a 7 or an 11. If his first toss is a 2, 3, or 12, he loses outright. If his first roll is something else, say, a 9, that number becomes his "point" and he keeps rolling the dice until he either rolls another 9, in which case he wins, or a 7, in which case he loses. Characterize the sample outcomes contained in the event "Shooter wins with a point of 9."

2.2.14. A probability-minded despot offers a convicted murderer a final chance to gain his release. The prisoner is given twenty chips, ten white and ten black. All twenty are to be placed into two urns, according to any allocation scheme the prisoner wishes, with the one proviso being that each urn contain at least one chip. The executioner will then pick one of the two urns at random and from that urn, one chip at random. If the chip selected is white, the prisoner will be set free; if it is black, he "buys the farm." Characterize the sample space describing the prisoner's possible allocation options. (Intuitively, which allocation affords the prisoner the greatest chance of survival?)

2.2.15. Suppose that ten chips, numbered 1 through 10, are put into an urn at one minute to midnight, and chip number 1 is quickly removed. At one-half minute to midnight, chips numbered 11 through 20 are added to the urn, and chip number 2 is quickly removed. Then at one-fourth minute to midnight, chips numbered 21 to 30 are added to the urn, and chip number 3 is quickly removed. If that procedure for adding chips to the urn continues, how many chips will be in the urn at midnight (148)?

Unions, Intersections, and Complements

Associated with events defined on a sample space are several operations collectively referred to as the *algebra of sets*. These are the rules that govern the ways in which one event can be combined with another. Consider, for example, the game of craps described in Question 2.2.13. The shooter wins on his initial roll if he throws either a 7 or an 11. In the language of the algebra of sets, the event "Shooter rolls a 7 *or* an 11" is the *union* of two simpler events, "Shooter rolls a 7" and "Shooter rolls an 11." If E denotes the union and if A and B denote the two events making up the union, we write $E = A \cup B$. The next several definitions and examples illustrate those portions of the algebra of sets that we will find particularly useful in the chapters ahead.

Definition 2.2.1. Let A and B be any two events defined over the same sample space S. Then

a. The *intersection* of A and B, written $A \cap B$, is the event whose outcomes belong to both A and B.
b. The *union* of A and B, written $A \cup B$, is the event whose outcomes belong to either A or B or both.

Example 2.2.6 A single card is drawn from a poker deck. Let A be the event that an ace is selected:

$$A = \{\text{ace of hearts, ace of diamonds, ace of clubs, ace of spades}\}$$

Let B be the event "Heart is drawn":

$$B = \{2 \text{ of hearts}, 3 \text{ of hearts}, \ldots, \text{ ace of hearts}\}$$

Then

$$A \cap B = \{\text{ace of hearts}\}$$

and

$$A \cup B = \{2 \text{ of hearts}, 3 \text{ of hearts}, \ldots, \text{ ace of hearts, ace of diamonds, ace of clubs, ace of spades}\}$$

(Let C be the event "Club is drawn." Which cards are in $B \cup C$? In $B \cap C$?) ■

Example 2.2.7 Let A be the set of x's for which $x^2 + 2x = 8$; let B be the set for which $x^2 + x = 6$. Find $A \cap B$ and $A \cup B$.

Since the first equation factors into $(x + 4)(x - 2) = 0$, its solution set is $A = \{-4, 2\}$. Similarly, the second equation can be written $(x + 3)(x - 2) = 0$, making $B = \{-3, 2\}$. Therefore,

$$A \cap B = \{2\}$$

and

$$A \cup B = \{-4, -3, 2\}$$ ■

Example 2.2.8 Consider the electrical circuit pictured in Figure 2.2.2. Let A_i denote the event that switch i fails to close, $i = 1, 2, 3, 4$. Let A be the event "Circuit is not completed." Express A in terms of the A_i's.

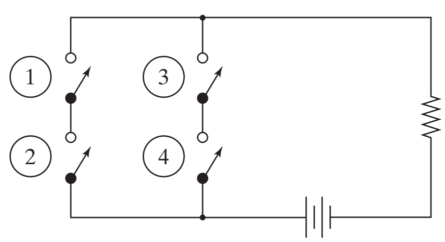


Figure 2.2.2

Call the ① and ② switches line a; call the ③ and ④ switches line b. By inspection, the circuit fails only if *both* line a and line b fail. But line a fails only if *either* ① *or* ② (or both) fail. That is, the event that line a fails is the union $A_1 \cup A_2$. Similarly, the failure of line b is the union $A_3 \cup A_4$. The event that the circuit fails, then, is an intersection:

$$A = (A_1 \cup A_2) \cap (A_3 \cup A_4)$$ ■

Definition 2.2.2. Events A and B defined over the same sample space are said to be *mutually exclusive* if they have no outcomes in common—that is, if $A \cap B = \emptyset$, where $\emptyset$ is the null set.

Example 2.2.9

Consider a single throw of two dice. Define A to be the event that the *sum* of the faces showing is odd. Let B be the event that the two faces themselves are odd. Then clearly, the intersection is empty, the sum of two odd numbers necessarily being even. In symbols, $A \cap B = \emptyset$. (Recall the event $B \cap C$ asked for in Example 2.2.6.) ■

Definition 2.2.3. Let A be any event defined on a sample space S. The *complement* of A, written A^C, is the event consisting of all the outcomes in S other than those contained in A.

Example 2.2.10

Let A be the set of (x, y)'s for which $x^2 + y^2 < 1$. Sketch the region in the xy-plane corresponding to A^C.

From analytic geometry, we recognize that $x^2 + y^2 < 1$ describes the interior of a circle of radius 1 centered at the origin. Figure 2.2.3 shows the complement—the points on the circumference of the circle and the points outside the circle.

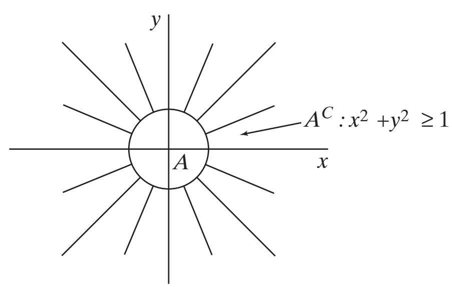


Figure 2.2.3 ■

The notions of union and intersection can easily be extended to more than two events. For example, the expression $A_1 \cup A_2 \cup \cdots \cup A_k$ defines the set of outcomes belonging to *any* of the A_i's (or to any combination of the A_i's). Similarly, $A_1 \cap A_2 \cap \cdots \cap A_k$ is the set of outcomes belonging to *all* of the A_i's.

Example 2.2.11

Suppose the events $A_1, A_2, \ldots, A_k$ are intervals of real numbers such that

$$A_i = \{x : 0 \leq x < 1/i\}, \quad i = 1, 2, \ldots, k$$

Describe the sets $A_1 \cup A_2 \cup \cdots \cup A_k = \cup_{i=1}^{k} A_i$ and $A_1 \cap A_2 \cap \cdots \cap A_k = \cap_{i=1}^{k} A_i$.

Notice that the A_i's are telescoping sets. That is, A_1 is the interval $0 \leq x < 1$, A_2 is the interval $0 \leq x < \frac{1}{2}$, and so on. It follows, then, that the *union* of the k A_i's is simply A_1 while the *intersection* of the A_i's (that is, their overlap) is A_k. ■

Questions

2.2.16. Sketch the regions in the xy-plane corresponding to $A \cup B$ and $A \cap B$ if

$$A = \{(x, y): 0 < x < 3, 0 < y < 3\}$$

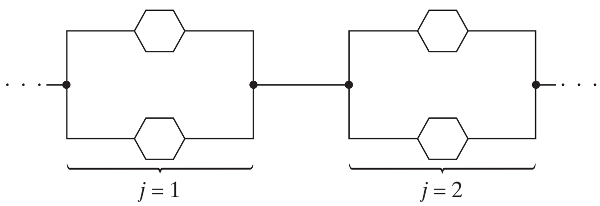
and

$$B = \{(x, y): 2 < x < 4, 2 < y < 4\}$$

2.2.17. Referring to Example 2.2.7, find $A \cap B$ and $A \cup B$ if the two equations were replaced by inequalities: $x^2 + 2x \leq 8$ and $x^2 + x \leq 6$.

2.2.18. Find $A \cap B \cap C$ if $A = \{x: 0 \leq x \leq 4\}$, $B = \{x: 2 \leq x \leq 6\}$, and $C = \{x: x = 0, 1, 2, \ldots\}$.

2.2.19. An electronic system has four components divided into two pairs. The two components of each pair are wired in parallel; the two pairs are wired in series. Let A_{ij} denote the event "ith component in jth pair fails," $i = 1, 2$; $j = 1, 2$. Let A be the event "System fails." Write A in terms of the A_{ij}'s.



2.2.20. Define $A = \{x : 0 \le x \le 1\}$, $B = \{x : 0 \le x \le 3\}$, and $C = \{x : -1 \le x \le 2\}$. Draw diagrams showing each of the following sets of points:

(a) $A^C \cap B \cap C$
(b) $A^C \cup (B \cap C)$
(c) $A \cap B \cap C^C$
(d) $[(A \cup B) \cap C^C]^C$

2.2.21. Let A be the set of five-card hands dealt from a 52-card poker deck, where the denominations of the five cards are all consecutive—for example, (7 of hearts, 8 of spades, 9 of spades, 10 of hearts, jack of diamonds). Let B be the set of five-card hands where the suits of the five cards are all the same. How many outcomes are in the event $A \cap B$?

2.2.22. Suppose that each of the twelve letters in the word

$$T\ E\ S\ S\ E\ L\ L\ A\ T\ I\ O\ N$$

is written on a chip. Define the events F, R, and C as follows:

F: letters in first half of alphabet
R: letters that are repeated
V: letters that are vowels

Which chips make up the following events?

(a) $F \cap R \cap V$
(b) $F^C \cap R \cap V^C$
(c) $F \cap R^C \cap V$

2.2.23. Let A, B, and C be any three events defined on a sample space S. Show that

(a) the outcomes in $A \cup (B \cap C)$ are the same as the outcomes in $(A \cup B) \cap (A \cup C)$.
(b) the outcomes in $A \cap (B \cup C)$ are the same as the outcomes in $(A \cap B) \cup (A \cap C)$.

2.2.24. Let $A_1, A_2, \ldots, A_k$ be any set of events defined on a sample space S. What outcomes belong to the event

$$(A_1 \cup A_2 \cup \cdots \cup A_k) \cup \left(A_1^C \cap A_2^C \cap \cdots \cap A_k^C\right)$$

2.2.25. Let A, B, and C be any three events defined on a sample space S. Show that the operations of union and intersection are *associative* by proving that

(a) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
(b) $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$

2.2.26. Suppose that three events—A, B, and C—are defined on a sample space S. Use the union, intersection, and complement operations to represent each of the following events:

(a) none of the three events occurs
(b) all three of the events occur
(c) only event A occurs
(d) exactly one event occurs
(e) exactly two events occur

2.2.27. What must be true of events A and B if

(a) $A \cup B = B$
(b) $A \cap B = A$

2.2.28. Let events A and B and sample space S be defined as the following intervals:

$$S = \{x : 0 \le x \le 10\}$$
$$A = \{x : 0 < x < 5\}$$
$$B = \{x : 3 \le x \le 7\}$$

Characterize the following events:

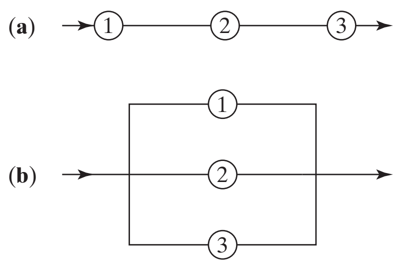
(a) A^C
(b) $A \cap B$
(c) $A \cup B$
(d) $A \cap B^C$
(e) $A^C \cup B$
(f) $A^C \cap B^C$

2.2.29. A coin is tossed four times and the resulting sequence of heads and/or tails is recorded. Define the events A, B, and C as follows:

A: exactly two heads appear
B: heads and tails alternate
C: first two tosses are heads

(a) Which events, if any, are mutually exclusive?
(b) Which events, if any, are subsets of other sets?

2.2.30. Pictured on the next page are two organizational charts describing the way upper management vets new proposals. For both models, three vice presidents—1, 2, and 3—each voice an opinion.



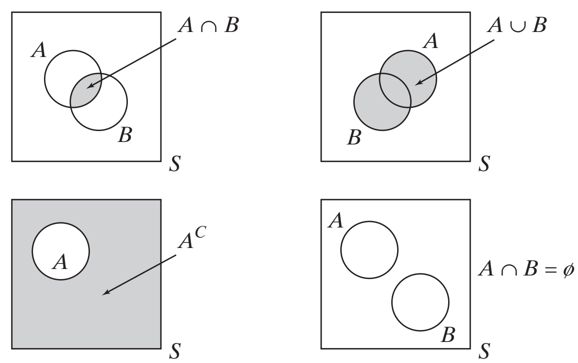
For (a), all three must concur if the proposal is to pass; if any one of the three favors the proposal in (b), it passes. Let A_i denote the event that vice president i favors the proposal, $i = 1, 2, 3$, and let A denote the event that the proposal passes. Express A in terms of the A_i's for the two office protocols. Under what sorts of situations might one system be preferable to the other?

Expressing Events Graphically: Venn Diagrams

Relationships based on two or more events can sometimes be difficult to express using only equations or verbal descriptions. An alternative approach that can be highly effective is to represent the underlying events graphically in a format known as a *Venn diagram*. Figure 2.2.4 shows Venn diagrams for an intersection, a union, a complement, and two events that are mutually exclusive. In each case, the shaded interior of a region corresponds to the desired event.

Figure 2.2.4

Venn diagrams



Example 2.2.12

When two events A and B are defined on a sample space, we will frequently need to consider

a. the event that *exactly one* (of the two) occurs.
b. the event that *at most one* (of the two) occurs.

Getting expressions for each of these is easy if we visualize the corresponding Venn diagrams.

The shaded area in Figure 2.2.5 represents the event E that either A or B, but not both, occurs (that is, *exactly one* occurs).

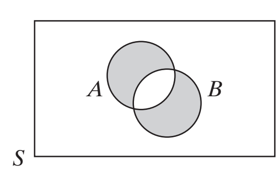


Figure 2.2.5

Just by looking at the diagram we can formulate an expression for E. The portion of A, for example, included in E is $A \cap B^C$. Similarly, the portion of B included in E is $B \cap A^C$. It follows that E can be written as a union:

$$E = (A \cap B^C) \cup (B \cap A^C)$$

(Convince yourself that an equivalent expression for E is $(A \cap B)^C \cap (A \cup B)$.)

Figure 2.2.6 shows the event F that *at most one* (of the two events) occurs. Since the latter includes every outcome except those belonging to *both* A and B, we can write

$$F = (A \cap B)^C$$

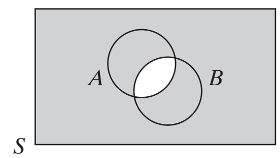


Figure 2.2.6

Questions

2.2.31. During orientation week, the latest Spiderman movie was shown twice at State University. Among the entering class of 6000 freshmen, 850 went to see it the first time, 690 the second time, while 4700 failed to see it either time. How many saw it twice?

2.2.32. Let A and B be any two events. Use Venn diagrams to show that

(a) the complement of their intersection is the union of their complements:

$$(A \cap B)^C = A^C \cup B^C$$

(b) the complement of their union is the intersection of their complements:

$$(A \cup B)^C = A^C \cap B^C$$

(These two results are known as *DeMorgan's laws*.)

2.2.33. Let A, B, and C be any three events. Use Venn diagrams to show that

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2.2.34. Let A, B, and C be any three events. Use Venn diagrams to show that

(a) $A \cup (B \cup C) = (A \cup B) \cup C$
(b) $A \cap (B \cap C) = (A \cap B) \cap C$

2.2.35. Let A and B be any two events defined on a sample space S. Which of the following sets are necessarily subsets of which other sets?

$$A \quad B \quad A \cup B \quad A \cap B \quad A^C \cap B$$
$$A \cap B^C \quad (A^C \cup B^C)^C$$

2.2.36. Use Venn diagrams to suggest an equivalent way of representing the following events:

(a) $(A \cap B^C)^C$
(b) $B \cup (A \cup B)^C$
(c) $A \cap (A \cap B)^C$

2.2.37. A total of twelve hundred graduates of State Tech have gotten into medical school in the past several years. Of that number, one thousand earned scores of twenty-seven or higher on the MCAT and four hundred had GPAs that were 3.5 or higher. Moreover, three hundred had MCATs that were twenty-seven or higher *and* GPAs that were 3.5 or higher. What proportion of those twelve hundred graduates got into medical school with an MCAT lower than twenty-seven and a GPA below 3.5?

2.2.38. Let A, B, and C be any three events defined on a sample space S. Let $N(A)$, $N(B)$, $N(C)$, $N(A \cap B)$, $N(A \cap C)$, $N(B \cap C)$, and $N(A \cap B \cap C)$ denote the numbers of outcomes in all the different intersections in which A, B, and C are involved. Use a Venn diagram to suggest a formula for $N(A \cup B \cup C)$. [*Hint:* Start with the