**Self-Test 1:**

1) $T_6(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6$
Estimate $\sin\left(\frac{2\pi}{3}\right)$: I compute $\frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$, so
$T_6\left(\frac{2\pi}{3}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{6}\right)^4 - \frac{1}{6!}\left(\frac{\pi}{6}\right)^6 \approx 0.866025264$
The actual value of $\sin\left(\frac{2\pi}{3}\right)$ is $\frac{\sqrt{3}}{2} \approx 0.866025404$
The estimate is smaller than the actual value by $\approx 0.000000140$. It agrees in the first 6 decimal places.

2) a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$ converges, because the sequence of absolute values converges to 0. $\{|a_n|\} = \{\frac{1}{n}\} \to 0$ as $n \to \infty$.

b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ converges, because the sequence is monotonic increasing and is bounded: $0 < \frac{n}{n+1} < 1$ for all n. A bounded monotonic sequence converges.

c) $\{a_n\} = \{\frac{n}{n^2-1}\}$ for n starting at 2 converges: $0 < \frac{n}{n^2-1} < \frac{n}{n^2} = \frac{1}{n} \to 0$ as $n \to \infty$.

d) $\{a_n\} = \{\frac{n+1}{n}\}$ converges: $\{\frac{n+1}{n}\} = \{1 + \frac{1}{n}\} \to 1$ as $n \to \infty$

3) a) $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \ldots$
$S_1 = 2$
$S_2 = 2 + \frac{2}{3} = \frac{8}{3}$
$S_3 = \frac{8}{3} + \frac{2}{9} = \frac{26}{9}$
$S_4 = \frac{26}{9} + \frac{2}{27} = \frac{236}{27}$
The series converges: it is a geometric series with $a = 2$ and $r = \frac{1}{3}$, so $|r| < 1$: it converges to $2\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{3}$

b) $5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + 1 + \frac{5}{6} + \ldots$
$S_1 = 5$
$S_2 = 5 + \frac{5}{2} = \frac{15}{2}$
$S_3 = \frac{15}{2} + \frac{5}{3} = \frac{55}{6}$
$S_4 = \frac{55}{6} + \frac{5}{4} = \frac{140}{12}$
This series can be rewritten as $5\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots\right)$
So it is 5 times the harmonic series, which diverges.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n}$
$S_1 = \frac{2}{3}$
$S_2 = \frac{2}{3} + \frac{1}{4} = \frac{11}{12}$
$S_3 = \frac{11}{12} + \frac{2}{15} = \frac{63}{60}$
$S_4 = \frac{63}{60} + \frac{1}{12} = \frac{69}{60}$
This series converges: $0 < \frac{2}{n^2+1} < \frac{2}{n^2}$, which is a multiple of a p-series with p=2, so it converges, but this does not tell us what the sum is.
Another way: we can rewrite the series by using partial fractions, and it becomes a telescoping series, so it converges. This also will give us the sum!
$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2 + 2n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$
$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots$ converges to 1.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ (Hint: this converges, but it's rather tricky to find the value of the sum.)
$S_1 = 1$
$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$
$S_3 = \frac{3}{2} + \frac{1}{6} = \frac{5}{3}$
$S_4 = \frac{5}{3} + \frac{1}{24} = \frac{41}{24}$

This series converges: $n! > n(n-1) = n^2 - n$ for all $n > 1$, so we can ignore the first term and compare the rest of the series with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ This converges by limit comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is a p-series with p=2. So the original series converges as well.
This does not tell us what the sum is, however. (The sum turns out to be $e$, and this will come from the Taylor series for the function $e^x$ later in the course.)

**Self-Test 2:**

1) $T_3(x) = -1 + \frac{x}{3} + \frac{x^2}{9} + \frac{5x^3}{81}$

To estimate $\sqrt[3]{-0.5}$ with this, we need to have $x - 1 = -0.5$ so $x = 0.5$
$T_3(0.5) = -1 + \frac{0.5}{3} + \frac{0.5^2}{9} + \frac{5(0.5^3)}{81} \approx -0.7978$
The actual value is $\sqrt[3]{-0.5} = (-0.5)^{\frac{1}{3}} \approx -0.7937$, so the estimate is too small by about 0.0041.

2) a) $\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \ldots$

This diverges: the terms oscillate from $\frac{1}{3}$ to $-\frac{1}{3}$, so they do not approach any limit.

b) $1, 1, 1, 1, 1, 1, \ldots$

This converges: the limit is 1.

c) $\{a_n\} = \{\frac{n+1}{n!}\}$

This converges: $\lim_{n\to\infty} \frac{n+1}{n!} = \lim_{n\to\infty} \frac{n}{n!} + \lim_{n\to\infty} \frac{1}{n!} = \lim_{n\to\infty} \frac{1}{(n-1)!} + 0 = 0$

d) $\{a_n\} = \{\frac{|\sin(n)|}{n}\}$

This converges: $\frac{|\sin(n)|}{n}$ is positive and $\frac{|\sin(n)|}{n} \leq \frac{1}{n}$ for all n, so $\lim_{n\to\infty} \frac{|\sin(n)|}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

3) a) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$

$S_1 = 1$
$S_2 = \frac{\sqrt{2}+1}{\sqrt{2}}$
$S_3 = \frac{\sqrt{6}+\sqrt{3}+\sqrt{2}}{\sqrt{6}}$
$S_4 = \frac{3\sqrt{6}+2\sqrt{3}+2\sqrt{2}}{2\sqrt{6}}$
The series diverges: it is a p-series with $p = \frac{1}{2}$.
You could also use integral comparison with $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}}$

b) $5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \frac{5}{16} + \ldots$

$S_1 = 5$
$S_2 = \frac{5}{2}$
$S_3 = \frac{15}{4}$
$S_4 = \frac{25}{8}$
The series converges: it is a geometric series $\sum_{i=0}^{\infty} ar^i$ with $a = 5$ and $r = -\frac{1}{2}$, so the sum is $5\left(\frac{1}{1-\left(-\frac{1}{2}\right)}\right) = \frac{10}{3}$

c) $\sum_{n=4}^{\infty} \frac{n}{n^2 - 2n - 3}$

$S_1 = \frac{4}{5}$
$S_2 = \frac{4}{5} + \frac{5}{12} = \frac{73}{60}$
$S_3 = \frac{73}{60} + \frac{2}{7} = \frac{631}{420}$
$S_4 = \frac{631}{420} + \frac{7}{32} = \frac{5883}{3360}$
This series diverges: you can use direct or limit comparison with $\sum_{n=4}^{\infty} \frac{1}{n}$, which diverges.

d) $\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$

$S_1 = \frac{1}{2}$
$S_2 = \frac{2}{3}$
$S_3 = \frac{3}{4}$
$S_4 = \frac{4}{5}$

This is a telescoping series, which you can see expanding using partial fractions: $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$, so it converges. The sum of the series is 1.