1) For the function $f(x) = \sqrt{x}$:

a) Find the 4th degree Taylor polynomial centered at 1 and simplify its coefficients as much as possible (no decimals)

$$\begin{array}{ll} f(x) = \sqrt(x) = x^{\frac{1}{2}} & f(1) = 1 \\ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} & f'(1) = \frac{1}{2} \\ f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} & f'''(1) = -\frac{1}{4} \\ f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} & f'''(1) = \frac{3}{8} \\ f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}} & f^{(4)}(1) = -\frac{15}{16} \end{array}$$

$$\begin{aligned} T_4(x) &= 1 + \frac{1}{2}(x-1) - \frac{1}{4} \cdot \frac{(x-1)^2}{2!} + \frac{3}{8} \cdot \frac{(x-1)^3}{3!} - \frac{15}{16} \cdot \frac{(x-1)^4}{4!} \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 \end{aligned}$$

b) Use that Taylor polynomial to estimate the value of $\sqrt{2}$ to 4 decimal places. How far away is your estimate from the true value of $\sqrt{2}$?

$T_4(2) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} \approx 1.3984$
$\sqrt{2} \approx 1.4142$, the difference is ≈ 0.0158.

2) Determine whether each of the following sequences is convergent or divergent. You must give an explanation for your conclusion in order to receive credit.

a) $\{a_n\} = \{\frac{n}{n+1}\}$

This sequence is convergent: $\lim_{n\to\infty} \left(\frac{n}{n+1}\right) = 1$

b) $\{a_n\} = \{(-1)^{n+1}\}$

This sequence is alternating 1 and -1: is is divergent.

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

This is the same sequence as in (a): it is convergent.
Alternatively: This sequence is monotonic increasing and is bounded between 0 and 1, so it is convergent.
[Note: this was an accident. I choose problems at random from my problem bank for the test, and accidentally wound up with the same sequence written twice in this version of the test. But you certainly should be able to handle this sequence no matter how it is written!]

3) For each of the following series, compute the partial sums S_1, S_2, and S_3. The determine whether the series is convergent or divergent. If it is convergent, find the sum.

a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

$S_1 = 0$
$S_2 = \frac{\ln(2)}{2} \approx 0.347$
$S_3 = \frac{\ln(2)}{2} + \frac{\ln(3)}{3} \approx 0.713$
This series is divergent: by direct comparison, for $n > 2$, $\ln(n) > 1$, so we have $\frac{\ln(n)}{n} > \frac{1}{n}$ for $n > 2$, and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges. (It is a harmonic series.)

b) $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \ldots$

$S_1 = 2$
$S_2 = \frac{4}{3}$
$S_3 = \frac{14}{9}$
This is a geometric series $\sum_{i=0}^{\infty} ar^i = a \sum_{i=0}^{\infty} r^i$ with $a = 2$ and $r = -\frac{1}{3}$. Since $|r| < 1$ it converges to $2\left(\frac{1}{1-(-\frac{1}{3})}\right) = \frac{3}{2}$

c) $\sum_{n=1}^{\infty} \frac{n}{n^2+3n+2}$

$S_1 = \frac{1}{6}$
$S_2 = \frac{1}{6} + \frac{2}{12} = \frac{1}{3}$
$S_3 = \frac{1}{3} + \frac{3}{20} = \frac{29}{60}$
This series is divergent by limit comparison with $\frac{1}{n}$: (Direct comparison cannot be used here.)

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{\frac{n}{n^2+3n+2}}{\frac{1}{n}}\right) &= \lim_{n\to\infty} \left(\frac{n}{n^2+3n+2} \cdot \frac{n}{1}\right) \\ &= \lim_{n\to\infty} \left(\frac{n^2}{n^2+3n+2}\right) = 1 > 0 \end{aligned}$$

c') (As I meant the problem to be)
$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

$S_1 = \frac{1}{6}$
$S_2 = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$
$S_3 = \frac{1}{4} + \frac{1}{20} = \frac{3}{10}$
This series is convergent: This can be seen by direct comparison with $\frac{1}{n^2}$, but that will not tell us the sum. The denominator factors into $(n+1)(n+2)$, and by using a partial fractions expansion we can see that this is a telescoping series:
$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

Solve to find that $A = 1, B = -1$, so the series is
$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots$ which is telescoping: the sum is $\frac{1}{2}$.