1) Find the indefinite integral $\int \sec^4(x)\mathrm{d}x$

$$\begin{aligned}\int \sec^4(x)\mathrm{d}x &= \int \left(\tan^2(x)+1\right)\sec^2(x)\mathrm{d}x \\ &= \int \tan^2(x)\sec^2(x)\mathrm{d}x + \int \sec^2(x)\mathrm{d}x \\ &= \frac{1}{3}\tan^3(x) + \tan(x) + C\end{aligned}$$

To get $\frac{1}{3}\tan^3(x)$ in the second line, use substitution $u = \tan(x) \implies \mathrm{d}u = \sec^2(x)\mathrm{d}x$. After awhile you should be able to do such substitutions in your head, but write it out if you need to.

2) Find the indefinite integral $\int \frac{1}{x^3+x}\mathrm{d}x$
The easiest way is to use partial fractions. It is also possible to use trig substitution.
$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$
$1 = A(x^2+1) + (Bx+C)x = Ax^2 + A + Bx^2 + Cx$
So $A+B=0$, $C=0$, $A=1 \implies B=-1$ from the first equation.

$$\begin{aligned}\int \frac{1}{x^3+x}\mathrm{d}x &= \int \frac{1}{x}\mathrm{d}x - \int \frac{x}{x^2+1}\mathrm{d}x \\ &= \ln|x| - \frac{1}{2}\int \frac{1}{u}\mathrm{d}u \\ &= \ln|x| - \frac{1}{2}\ln|u| + C \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + C\end{aligned}$$

In the second line we used the substitution $u = x^2+1 \implies \mathrm{d}u = 2x\mathrm{d}x$.
The final result could also be written (using a property of logarithms) as $\ln|x| - \ln\sqrt{x^2+1} + C$

3) Find the indefinite integral $\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x$
Use the trig substitution $x = \sec\theta \implies \mathrm{d}x = \tan\theta\sec\theta\mathrm{d}\theta$

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x &= \int \frac{\sec^2\theta\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\mathrm{d}\theta \\ &= \int \frac{\sec^3\theta\tan\theta}{\tan\theta}\mathrm{d}\theta \\ &= \int \sec^3\theta\mathrm{d}\theta\end{aligned}$$

Now either you may recall what that integral was from the example worked in class, or you will work it out here:

$$\begin{aligned}\int \sec^3\theta\mathrm{d}\theta &= \int \left(tan^2\theta - 1\right)\sec\theta\mathrm{d}\theta \\ &= \int \tan^2\theta\sec\theta\mathrm{d}\theta - \int \sec\theta\mathrm{d}\theta\end{aligned}$$

In the first integral, use integration by parts with $u = \tan\theta \implies \mathrm{d}u = \sec^2\theta \mathrm{d}\theta$ and $\mathrm{d}v = \tan\theta\sec\theta\mathrm{d}\theta \implies v = \sec\theta$. We know the second integral from previous work.

$$\begin{aligned}\int \sec^3\theta \mathrm{d}\theta &= \int \tan^2\theta\sec\theta\mathrm{d}\theta - \int \sec\theta\mathrm{d}\theta \\ &= \tan\theta\sec\theta - \int \sec^3\theta\mathrm{d}\theta + \int\sec\theta\mathrm{d}\theta\end{aligned}$$

Now $\int \sec^3\theta\mathrm{d}\theta$ appears on both sides of this equation so we move it from the right-hand side to get:

$$2\int \sec^3\theta\mathrm{d}\theta = \tan\theta\sec\theta + \ln|\sec\theta + \tan\theta| + C$$

So $\int \sec^3\theta\mathrm{d}\theta = \frac{1}{2}\left(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|\right) + C$

Now substitute back using $x = \sec\theta$ and $\tan\theta = \sqrt{x^2-1}$:
Finally we get $\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln|x+\sqrt{x^2-1}|$

4) Find the value of the definite integral $\int_{-1}^{2}\frac{1}{x^2-4}\mathrm{d}x$ in exact form and then round to the nearest thousandth.
This integral will be an improper integral, since the denominator is 0 when $x = 2$ [That was not my intention: I did not notice that I had not entered the bounds of integration correctly. But we can deal with that anyway.]
$\int_{-1}^{2}\frac{1}{x^2-4}\mathrm{d}x = \lim_{b\to 2^-}\int_{-1}^{b}\frac{1}{x^2-4}\mathrm{d}x$
The easiest way to find the integral is to use partial fractions. It is also possible to use trig substitution.
$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$
$1 = A(x+2) + B(x-2) = Ax + 2A + Bx - 2B = (A+B)x + (2A-2B)$
So $A + B = 0$ and $2A - 2B = 1$
Solve that system to find $A = \frac{1}{4}$, $B = -\frac{1}{4}$.

$$\begin{aligned}\int_{-1}^{b}\frac{1}{x^2-4}\mathrm{d}x &= \frac{1}{4}\int_{-1}^{b}\frac{1}{x-2}\mathrm{d}x - \frac{1}{4}\int_{-1}^{b}\frac{1}{x+2}\mathrm{d}x \\ &= \frac{1}{4}\left[\ln|x-2| - \ln|x+2|\right]_{-1}^{b} \\ &= \frac{1}{4}(\ln|b-2| - \ln|b+2| - \ln(3) + \ln(1))\end{aligned}$$

Here we used the obvious substitutions in the integrals to get the logarithms.
Now take the limit:
$\lim_{b\to 2^-}\frac{1}{4}(\ln|b-2| - \ln|b+2| - \ln(3) + \ln(1)) = \frac{1}{4}(\lim_{b\to 2^-}\ln|b-2| - \ln(4) - \ln(3))$
and $\lim_{b\to 2^-}\ln|b-2|$ does not exist (it diverges to $-\infty$), so this integral diverges.

It would be good practice to do the integral I intended to have here:
$\int_{-1}^{1.5}\frac{1}{x^2-4}\mathrm{d}x = \frac{1}{4}(\ln(0.5) - \ln(3.5) - \ln(3) + \ln(1)) = \frac{1}{4}\ln\left(\frac{1}{21}\right) \approx -0.761$ to the nearest thousandth.

5) Find the value of $\displaystyle\int_4^8 \frac{\mathrm{d}x}{\sqrt{x-4}}$ or else show that it diverges.

This is an improper integral, and you must show that you know what the definition of the improper integral is when you work it out. It is true that in the end we will find the limit by substitution, but that is far from obvious at the beginning. If you are not careful about using the limit definition, you can get into a lot of trouble: see Example 195 part 2 in the textbook (p. 337) for an instance of this! (I did discuss it in class for this reason!)

$$\int_4^8 \frac{\mathrm{d}x}{\sqrt{x-4}} = \lim_{a\to 4^+}\int_a^8 \frac{\mathrm{d}x}{\sqrt{x-4}}$$

Use the substitution $u = x-4 \implies \mathrm{d}u = \mathrm{d}x$ to find this:

$$\begin{aligned}\int_a^8 \frac{\mathrm{d}x}{\sqrt{x-4}} &= \int_{a-4}^4 \frac{\mathrm{d}u}{\sqrt{u}}\\ &= \int_{a-4}^4 u^{-1/2}\mathrm{d}u\\ &= \left[2u^{1/2}\right]_{a-4}^4\\ &= 2\sqrt{4} - 2\sqrt{a-4} = 4 - 2\sqrt{a-4}\end{aligned}$$

Now take the limit: $\displaystyle\lim_{a\to 4^+}\int_a^8 \frac{\mathrm{d}x}{\sqrt{x-4}} = \lim_{a\to 4^+}(4-2\sqrt{a-4}) = 4-0 = 4$

Note: the only reason we can use substitution to find the limit here is that it is a one-sided limit and the function is continuous from the right. That is why you need to be careful about taking the limits in computing this kind of integral. If the limit had been from the left or two-sided, it would not exist. Make sure that you understand this.

6) Use a comparison test to determine whether $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^3+6x}}$ converges or not.

Looking at the leading term in the denominator of $f(x) = \frac{1}{\sqrt{x^3+6x}}$ suggests using a comparison with $g(x) = \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$, whose integral converges ($p > 1$).

Check if we can use direct comparison, which for convergence would require that $0 < f(x) < g(x)$:

When $x \geq 1$, $\sqrt{x^3+6x} > \sqrt{x^3}$, so $\frac{1}{\sqrt{x^3+6x}} < \frac{1}{\sqrt{x^3}}$ as needed.

By direct comparison, since $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^{3/2}}$ converges, so does $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^3+6x}}$.