This is to show that the series $\sum_{n=0}^{\infty} ne^{-x^2}$ converges.

We would like to compare with the integral

$$\int_0^\infty xe^{-x^2}\mathrm{d}x$$

,

but notice that the integral convergence test requires that the function we use for integrating must be positive and **decreasing** on the interval of integration. The function $f(x) = xe^{-x^2}$ is initially increasing on $[0, \infty)$ but by looking at the graph we can see that eventually it is decreasing, say after x=2. It makes no change to the convergence or divergence of a series if we leave off some terms at the beginning - only the "tail" of the series determines whether or not it converges. So we'll start the sum at n=2 and the integral at x=2:

$$\begin{aligned}\int_2^\infty xe^{-x^2}\mathrm{d}x &= \lim_{b\to\infty}\int_2^b xe^{-x^2}\mathrm{d}x\\ &= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_2^b \text{ after integrating by substitution } u = e^{-x^2}\\ &= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2^2} + \frac{1}{2}e^{-b^2}\right]\\ &= \left[-\frac{1}{2}e^{-4} + 0\right]\end{aligned}$$

So the integral converges: this means that our series converges as well.